On Wegner's inequality for axis-parallel rectangles

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Abstract

According to an old conjecture of Wegner, the piercing number of a set of axis-parallel rectangles in the plane is at most twice the independence number (or matching number) minus 1, that is, $\tau(\mathcal{F}) \leq 2\nu(\mathcal{F}) - 1$. On the other hand, the current best upper bound, due to Corea et al. (2015), is a $O\left((\log\log \nu(\mathcal{F}))^2\right)$ factor away from the current best lower bound. From the other direction, lower bound constructions with $\tau(\mathcal{F}) \geq 2\nu(\mathcal{F}) - 4$ are known. Here we exhibit families of such rectangles with $\tau = 7$ and $\nu = 4$ and thereby show that Wegner's inequality, if true, cannot be improved for $\nu = 4$. The analogous result for $\nu = 3$, due to Wegner, dates back to 1968.

A key element in our proof is establishing a connection with the Maximum Empty Box problem: Given a set P of n points inside an axis-parallel box U in $\mathbb{R}^d$, find a maximum-volume axis-parallel box that is contained in U but contains no points of P in its interior.

Whereas our construction can be extended to any larger independence number ($\nu = 5, 6, \ldots$) its analysis remains open.

Keywords: Piercing number, matching number, (p, q)-property, axis-parallel rectangle, largest empty box.

1 Introduction

Given a collection of sets E, a *piercing set* is a set of elements from $\cup_{F \in E} F$ intersecting every set in E. The *piercing number* of E is the minimal size of a piercing set. Given a hypergraph $H = (X, E)$, a *cover* of H is a set $C \subseteq X$ such that every edge of H contains a point in C, namely, for every $e \in E$ we have $e \cap C \neq \emptyset$. As such, a cover is precisely a piercing set of E. The *piercing number* $\tau(H)$ of a hypergraph $H = (X, E)$ is the piercing number of its edge set E. It is sometimes also called the *covering number* or *stabbing number* of the hypergraph.

Given integers $p \geq q > 1$, a family $\mathcal{F}$ of sets is said to satisfy the (p, q)*-property* if among every p sets in $\mathcal{F}$ there exist q sets with a non-empty intersection. The *independence number* or *matching number* of $\mathcal{F}$, namely the maximum number of pairwise disjoint sets in $\mathcal{F}$, is denoted by $\alpha(\mathcal{F})$ or $\nu(\mathcal{F})$. Clearly, $\nu(\mathcal{F}) \leq \tau(\mathcal{F})$. If $\nu(\mathcal{F}) = 1$ then we say that $\mathcal{F}$ is an *intersecting* family.

In the above terminology, Helly's theorem [22] says that if a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ satisfies the $(d + 1, d + 1)$-property then $\tau(\mathcal{F}) = 1$. Finding the piercing number of families of sets in $\mathbb{R}^d$ satisfying the (p, q)-property has been known in the literature as the (p, q)*-problem*. In particular, a collection of pairwise-intersecting intervals (i.e., an intersecting family of intervals) must have a point that belongs to all the intervals.

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Hadwiger and Debrunner [20, 21] conjectured in 1957 that the (p, q)-property in a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ implies that $\tau(\mathcal{F})$ is bounded by a constant depending on d, p, and q. They proved this under the condition that $(d-1)p < d(q-1)$ in the following stronger form:

Theorem 1 (Hadwiger–Debrunner [20]). *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ satisfying the (p, q)-property for $p \geq q > 1$. If $(d-1)p < d(q-1)$ then $\tau(\mathcal{F}) \leq p - q + 1$.*

In 1992 Alon and Kleitman [3] resolved the Hadwiger–Debrunner conjecture, proving that in a family of convex sets in $\mathbb{R}^d$ that satisfies the (p, q)-property, the piercing number is bounded by a constant:

Theorem 2 (Alon–Kleitman [3]). *Let $p \geq q \geq d+1$ be integers. Then there exists a constant $c = c(d; p, q)$ depending only on d, p, q, such that if a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ satisfies the (p, q)-property then $\tau(\mathcal{F}) \leq c$.*

In many cases the upper bounds on the piercing number improve significantly if we deal with families of "nice" sets. One such example is a result by Danzer, who proved:

Theorem 3 (Danzer [10]). *If a family of disks in $\mathbb{R}^2$ satisfies the $(2, 2)$-property, then $\tau(\mathcal{F}) \leq 4$.*

In this paper we restrict our attention to axis-parallel *hyper-rectangles* (or *boxes*) in $\mathbb{R}^d$. The following inequality [16, Ineq. (3.4), p. 355] applies to any such family $\mathcal{F}$ that satisfies the (p, q)-property:

$$\tau(\mathcal{F}) \leq \binom{p-q+d}{d}, \quad p \geq q \geq 2. \tag{1}$$

Many have examined the case $q = 2$. The main unsettled question here is whether $\tau(\mathcal{F}) = O(\nu(\mathcal{F}))$. The following is a long-standing conjecture in dimension 2:

Conjecture 1 (Wegner [32], Gýarfás–Lehel [19]). *If a family $\mathcal{F}$ of axis-parallel rectangles in $\mathbb{R}^2$ satisfies the $(p, 2)$-property, then $\tau(\mathcal{F}) \leq 2p - 3$.*

The $(p, 2)$-property can be rephrased as a family $\mathcal{F}$ of axis-parallel rectangles in $\mathbb{R}^2$ with $\nu(\mathcal{F}) = p - 1$. As such, Conjecture 1 can be formulated as follows:

Conjecture 2. *If $\mathcal{F}$ is a family of axis-parallel rectangles in $\mathbb{R}^2$, then $\tau(\mathcal{F}) \leq 2\nu(\mathcal{F}) - 1$.*

Károlyi [26] proved that if $\mathcal{F}$ is a family of axis-parallel boxes in $\mathbb{R}^d$, then

$$\tau(\mathcal{F}) \leq \nu(\mathcal{F})(1 + \log(\nu(\mathcal{F})))^{d-1}. \tag{2}$$

For the planar case, Eckhoff [16] gives the following upper-bound inequality based on a recurrence relation found independently by Wegner [33] and by Fon-Der-Flaass and Kostochka [17].

$$\tau(\mathcal{F}) \leq (\nu(\mathcal{F}) + 1)\lceil \log(\nu(\mathcal{F}) + 1) \rceil - 2^{\lceil \log(\nu(\mathcal{F})+1) \rceil} + 1. \tag{3}$$

After about 25 years, Correa et al. [9] improved Károlyi's bound for the plane by combining results of [4] and [6].

$$\tau(\mathcal{F}) = O\left(\nu(\mathcal{F}) \cdot (\log \log \nu(\mathcal{F}))^2\right). \tag{4}$$

From the other direction, Jelínek found an elegant construction with $\tau(\mathcal{F}) = 2\nu(\mathcal{F}) - 4$, for every $\nu \geq 4$ [9], and thereby showed that the factor 2 in Wegner's conjecture cannot be improved. On the other hand, one may note that this bound is not competitive for small ν, e.g., $\nu = 4, 5$. Our Theorem 4 below is relevant in this case.

For the special case of squares, better bounds are in effect. It is known that $\tau(\mathcal{F}) \leq 4\nu(\mathcal{F})$ for families of squares and $\tau(\mathcal{F}) \leq 2\nu(\mathcal{F}) - 1$ for families of unit squares [1, 12, 13]. The current state of the art for the ratio $\limsup \tau(\mathcal{F})/\nu(\mathcal{F})$ depending on the rectangle-type in the family is summarized in Table 1.

	Rectangles	Squares	Unit squares
Upper bound	$O\left((\log\log\nu)^2\right)$	4	2
Lower bound	2	3/2	3/2

Table 1: Bounds on the ratio $\limsup \tau(\mathcal{F})/\nu(\mathcal{F})$.

The function $f(n)$. In order to study the dependence between $\tau(\mathcal{F})$ and $\nu(\mathcal{F})$ it is convenient to define an integer function. As in [17], define $f(n)$ as the minimum integer such that every family $\mathcal{F}$ of axis-parallel rectangles with $\nu(\mathcal{F}) \leq n$ can be pierced by $f(n)$ points. (Alternatively, the condition $\nu(\mathcal{F}) \leq n$ can be replaced by $\nu(\mathcal{F}) = n$ in the definition.) A line sweep argument found independently by Wegner [33] and Fon-Der-Flaass and Kostochka [17] (mentioned previously; see also [16, Ineq. (3.6), p. 356]) yields the following recurrence:

$$f(n) \leq f\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n-1}{2} \right\rceil\right) + n. \tag{5}$$

Taking into account that $f(0) = 0$ and $f(1) = 1$, one immediately obtains

$$\begin{aligned}
f(2) &\leq f(0) + f(1) + 2 = 3,\\
f(3) &\leq f(1) + f(1) + 3 = 5,\\
f(4) &\leq f(1) + f(2) + 4 \leq 8,\\
f(5) &\leq f(2) + f(2) + 5 \leq 11,\\
&\vdots
\end{aligned}$$

The general solution to the recurrence (5) (equivalent to (3)) is:

$$f(n) \leq (n+1)\lceil \log(n+1) \rceil - 2^{\lceil \log(n+1) \rceil} + 1. \tag{6}$$

Lower bound constructions yield $f(1) = 1$, $f(2) = 3$, and $f(3) = 5$, and these are the only exact values known [16]. For small n, the resulting bounds are recorded in Table 2.

n	2	3	4	5
Upper bound on $f(n)$	3	5	8	11
Lower bound on $f(n)$	3	5	7(∗)	8(∗)

Table 2: Bounds on $f(n)$ for small n. The starred entries are proved in this paper.

Our results. (i) Our main result—the first starred entry in Table 2—is Theorem 4 below (its proof appears in Section 4). It gives $f(4) \geq 7$; recall that $f(4) \leq 8$ is known. (ii) A lower bound on the ratio $\tau(\mathcal{F})/\nu(\mathcal{F})$ in higher dimensions is given by Theorem 5 in Section 2 (its proof appears in Section 5). Both results rely on the connection between piercing numbers for families of axis-parallel boxes in $\mathbb{R}^d$ and the Maximum Empty Box problem in $[0,1]^d$, introduced in Section 2.

Theorem 4. *There exists a finite family $\mathcal{S}$ of axis-parallel rectangles with $\nu(\mathcal{S}) = 4$ and $\tau(\mathcal{S}) = 7$. That is, $f(4) \geq 7$.*

Related work. Among the many variants of the (p, q)-property and Helly's theorem in particular, we can only mention a few here. Danzer and Grünbaum [11] investigated the following problem: if d and n are positive integers, what is the smallest $h = h(d, n)$ such that a family of boxes in $\mathbb{R}^d$ is n-pierceable if each of its h-member subfamilies is n-pierceable? The showed that $h(d, n)$ is infinite for all (d, n) with $d \geq 2$ and $n \geq 3$ except for $(d, n) = (2, 3)$ when it is 16.

Larman et al. [29] showed that any collection of n axis-parallel rectangles contains $\sqrt{n/\log n}$ of them which are pairwise intersecting or pairwise disjoint; on the other hand, there are trivial examples with at most $\sqrt{n}$ in each of the two classes. If the conjecture $\tau(\mathcal{F}) = O(\nu(\mathcal{F}))$ were true, then the lower bound $\sqrt{n/\log n}$ could be improved to $\Omega(\sqrt{n})$; see also [5, p. 410].

Hadwiger had asked whether any collection of closed convex sets where every four have a triple that has a nonempty intersection (i.e., has at least one point in common) can be pierced by two points. Danzer exhibited six congruent triangles in the plane that can only be pierced by three points. The current best bound on the piercing number for such a family with the $(4, 3)$-property is 13; this bound is due to Kleitman et al. [28]. As such, the current gap for this problem is between 3 and 13.

Károlyi and Tardos [27] studied transversal numbers of hypergraphs related to multiple intervals and axis-parallel rectangles. Kaiser and Rabinovich [24] formulated a multicomponent generalization of Helly's theorem to convex (n, d)-bodies. Karasev [25] considered the problem of piercing families of convex sets in $\mathbb{R}^d$ such that every d or fewer sets in the family have a common point. Chan and Har-Peled [7] proved that for every family $\mathcal{F}$ of axis-parallel rectangles in $\mathbb{R}^2$ in which for every two intersecting rectangles, one of them contains a corner of the other, we have $\tau(\mathcal{F}) = O(\nu(\mathcal{F}))$.

Aronov, Ezra, and Sharir [4] have studied the of size of ε-nets for axis-parallel rectangles and boxes. Chalermsook and Chuzhoy [6] gave a $O(\log \log n)$-approximation algorithm for the problem of computing a Maximum Independent Set of Rectangles (MISR). Correa et al. [9] have used the above-mentioned results in combination. Besides combinatorial results, they have obtained several approximation algorithms for piercing various classes of rectangles, e.g., diagonal-pierced rectangles.

Govindarajan and Nivasch [18] studied a strengthening of the (p, q)-property by requiring that, among every p members of $\mathcal{S}$, at least q meet at a point of X, where X is a fixed convex curve in the plane; they showed that the piercing number can be substantially reduced in that case. Chudnovsky, Spirkl, and Zerbib [8] showed that if for each two intersecting boxes in a family $\mathcal{F}$ of boxes in $\mathbb{R}^d$, a corner of one is contained in the other, then $\mathcal{F}$ can be pierced by at most $O(k \log \log k)$ points, where $k = \nu(\mathcal{F})$, and in the special case where $\mathcal{F}$ contains only cubes this bound improves to $O(k)$.

Su and Zerbib [31] recently showed that results on piercing numbers have a natural interpretation in voting theory. For a survey of piercing in the context of geometric transversals, the reader is referred to the survey article [23].

2 Setup

In this section, we demonstrate an idea of constructing lower bound examples (i.e., families of axis-parallel rectangles) that support Wegner's Conjecture 2. By applying this idea, examples with $\tau(\mathcal{F}) = 2\nu(\mathcal{F}) - 1$ for $\nu(\mathcal{F}) = 2, 3, 4$ are obtained respectively. The last result in this sequence proves Theorem 4 (in Section 4). Whereas examples for the previous two ratios were previously known, we include ours to help the reader understand the construction better; as it illustrates the main ideas at a smaller scale.

2.1 Maximum Empty Box

A *box* in $\mathbb{R}^d$, $d \geq 2$, is a closed axis-parallel hyperrectangle $[a_1, b_1] \times \cdots \times [a_d, b_d]$ with $a_i \leq b_i$ for $1 \leq i \leq d$. Given a set S of n points in the unit cube $U_d = [0,1]^d$, a box $B \subset U_d$ is *empty* if it contains no points of S in its interior. Let $A_d(S)$ be the maximum volume of an empty box contained in U_d, and let $A_d(n)$ be the minimum value of $A_d(S)$ over all sets S of n points in U_d.

For a fixed d, it is known [30] that $A_d(n)$ is of the order $\Theta(\frac{1}{n})$. The following upper bound holds for any $d \geq 2$:

$$A_d(n) < \left(2^{d-1} \prod_{i=1}^{d-1} p_i \right) \cdot \frac{1}{n}, \tag{7}$$

where p_i is the ith prime, as shown in [30, 14]. In particular, $A_2(n) < \frac{4}{n}$.

A sharper upper bound has been recently obtained for larger d. The current best upper and lower bounds on $A_d(n)$, for $d \geq 54$, are as follows:

$$\frac{\log d}{4(n + \log d)} \leq A_d(n) \leq \frac{2^{7d+1}}{n}. \tag{8}$$

The lower bound is due to Aistleitner, Hinrichs, and Rudolf [2], and the upper bound is due to Larcher [2, Section 4]. For $d = 2$ the current best bounds (see [14, 15]) are

$$\big(5A_2(4) - o(1)\big) \cdot \frac{1}{n} = \big(1.25 - o(1)\big) \cdot \frac{1}{n} \leq A_2(n) < \frac{4}{n}. \tag{9}$$

Following Aistleitner et al. [2], define

$$c_d = \liminf_{n \to \infty} n \cdot A_d(n). \tag{10}$$

Taking (8) into account, we have

$$\frac{\log d}{4} \leq c_d \leq 2^{7d+1}, \text{ for } d \geq 2. \tag{11}$$

2.2 Discretization and connection with Maximum Empty Box

A long-standing open question—appearing in Eckhoff's survey [16, p. 359]—is whether $\tau = O(\nu)$ for systems of axis-parallel boxes in a fixed dimension d. Whereas we cannot answer this question, here we show that the ratio τ/ν must grow with the dimension d and further elaborate on the rate of this growth. It follows from (1) that $\tau(\mathcal{F}) \leq d + 1$, for any family of boxes in $\mathbb{R}^d$ having the $(3,2)$-property. In particular, $\tau(\mathcal{R}) \leq d+1$, for systems of axis-parallel boxes in $\mathbb{R}^d$ with $\nu(\mathcal{R}) = 2$. On the other hand, we have

$$\tau(\mathcal{R})/\nu(\mathcal{R}) = \Omega(\sqrt{d}/\log d) \tag{12}$$

for systems of axis-parallel boxes in $\mathbb{R}^d$ with $\nu(\mathcal{R}) = 2$ (this can be derived from a classical result of Erdős on k-chromatic triangle-free graphs, see [17, 16]). By taking multiple copies of this construction, it follows that there exist families of axis-parallel boxes in $\mathbb{R}^d$ with any given ν for which (12) holds.

One may wonder if there is any relation between (11) and (12). Observe that the large gap in (11) for the key parameter c_d leaves plenty of room for improvement. We next show that if $c_d = \omega(\sqrt{d}/\log d)$ were to hold, then one would obtain systems of axis-parallel boxes in $\mathbb{R}^d$ with $\tau(\mathcal{R})/\nu(\mathcal{R}) = \omega(\sqrt{d}/\log d)$, and thereby improve the lower bound in (12). The following result can be derived from Lemma 3 in combination with the aforementioned result of Aistleitner et al. [2].

Theorem 5. *For every $d \geq 1024$, there exists a system of axis-parallel boxes in $\mathbb{R}^d$ where $\tau(\mathcal{R})/\nu(\mathcal{R}) \geq c_d/2$.*

In particular, by the current state of the art, we have $\tau(\mathcal{R})/\nu(\mathcal{R}) = \Omega(\log d)$, a bound that grows with d but is inferior to the bound in (12).

We start with a technical lemma that provides a discretization mechanism for extracting a *finite* family of hyper-rectangles from an *infinite* family. For a finite point set $P \subset U_d$, $a, \delta > 0$, where $A_d(P) \geq a + 2\delta$, $a + 2\delta < 1$, and $1/\delta \in \mathbb{N}$, let $\mathcal{R}'(P, a, \delta; d)$ denote the *infinite* family of axis-parallel empty boxes of volume at least $a + 2\delta$ in U_d. Observe that if $P \subset P'$, then $\mathcal{R}'(P', a, \delta; d) \subset \mathcal{R}'(P, a, \delta; d)$.

Lemma 1. *For $P \subset U_d$, $a, \delta > 0$, where $A_d(P) \geq a + 2\delta$, $a + 2\delta < 1$, and $1/\delta \in \mathbb{N}$, there exists a finite family of axis-parallel empty boxes in U_d, denoted by $\mathcal{R}(P, a, \delta; d)$, so that:*

(i) *for each box $r \in \mathcal{R}(P, a, \delta; d)$ we have $r \cap P = \emptyset$ and $\mathrm{Vol}(r) = a + \delta$,*

(ii) *for every $r' \in \mathcal{R}'(P, a, \delta; d)$, there exists $r \in \mathcal{R}(P, a, \delta; d)$, with $r \subseteq r'$.*

(In particular, every box in $\mathcal{R}(P, a, \delta; d)$ has no points of P on its boundary.)

Proof. Let $j = 4d/\delta + 1$, and consider the $j \times \cdots \times j$ d-dimensional grid contained in U_d:

$$x_i = \frac{0}{j-1}, \frac{1}{j-1}, \ldots, 1, \text{ for } i = 1, \ldots, d.$$

Let $\mathcal{R}_1$ be the set of non-degenerate boxes determined by this grid. Note that $\mathcal{R}_1$ is a *finite* set whose cardinality is $|\mathcal{R}_1| = \binom{j}{2}^d$. Let $\mathcal{R}_2 \subset \mathcal{R}_1$ be the subset of grid boxes with volume at least $a + 1.5\delta$. Let $\mathcal{R}_3$ be the set of concentric scaled down homothetic copies of boxes in $\mathcal{R}_2$ of volume exactly $a+\delta$. Observe that $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ are independent of P. Set $\mathcal{R}(P, a, \delta; d) = \{r \in \mathcal{R}_3 \mid r \cap P = \emptyset\}$. It is not clear a priori, whether this set is non-empty; we argue below that it is.

Consider any hyper-rectangle $r' \in \mathcal{R}'(P, a, \delta; d)$; assume that $r' = [s_1, t_1] \times \cdots \times [s_d, t_d] \subseteq U_d$, where $\mathrm{Vol}(r') \geq a + 2\delta$. Put $\Delta_j = t_j - s_j$ for $j = 1, \ldots, d$; we have $\mathrm{Vol}(r') = \prod_{j=1}^d \Delta_j$ and

$$a + 2\delta \leq \Delta_j \leq 1, \text{ for } j = 1, \ldots, d.$$

By construction, each interval $[s_j, t_j]$ contains a *grid*-interval I_j of length

$$|I_j| \geq \Delta_j - \frac{2}{j-1} = \Delta_j - \frac{2\delta}{4d} = \Delta_j - \frac{\delta}{2d}.$$

Note that

$$|I_j| \geq a + 2\delta - \frac{\delta}{2d} \geq a + 1.75\delta.$$

Then $r_2 := \prod_{j=1}^d I_j \in \mathcal{R}_1$ is a *grid* hyper-rectangle (box) whose volume is

$$\begin{aligned} \mathrm{Vol}(r_2) &\geq \prod_{j=1}^d \left(\Delta_j - \frac{\delta}{2d}\right) \geq \prod_{j=1}^d \Delta_j - \sum_{j=1}^d \frac{\delta}{2d} \\ &= \prod_{j=1}^d \Delta_j - \frac{\delta}{2} \geq a + 2\delta - 0.5\,\delta = a + 1.5\delta. \end{aligned} \tag{13}$$

The first inequality above follows from Lemma 2 below with $k = d$, $a_i = \Delta_i$, and $\delta_i = \delta/(2d)$ for $i = 1, \ldots, k$. This implies $r_2 \in \mathcal{R}_2$. Furthermore, since r_2 is contained in r', it is empty of points of P in its interior.

By construction, the smaller concentric homothetic copy of r_2 of volume exactly $a+\delta$, denoted here by r, belongs to $\mathcal{R}_3$. Since r lies strictly inside r_2, we have $r \cap P = \emptyset$ and thus $r \in \mathcal{R}(P, a, \delta; d)$ as required. □

Lemma 2. *Let $a, \delta \in (0,1)$, where $a + 2\delta \leq \prod_{i=1}^{k} a_i \leq a_1, \ldots, a_k \leq 1$ and $\delta_1, \ldots, \delta_k > 0$, where $\sum_{i=1}^{k} \delta_i \leq \delta$. Then*

$$\prod_{i=1}^{k}(a_i - \delta_i) \geq \prod_{i=1}^{k} a_i - \sum_{i=1}^{k} \delta_i.$$

Proof. By the hypothesis, we have $a_i \geq a + 2\delta > \delta > \delta_i$, for every i. We prove the inequality by induction on k. For $k = 1$ there is nothing to prove. Assume that the inequality holds for $k-1$:

$$\prod_{i=1}^{k-1}(a_i - \delta_i) \geq \prod_{i=1}^{k-1} a_i - \sum_{i=1}^{k-1} \delta_i. \tag{14}$$

The left hand-side of (14) is clearly positive. Observe that

$$\prod_{i=1}^{k-1} a_i - \sum_{i=1}^{k-1} \delta_i \geq \prod_{i=1}^{k} a_i - \sum_{i=1}^{k} \delta_i \geq a + 2\delta - \delta = a + \delta,$$

thus the right hand-side of (14) is also positive. We can multiply the inequality by $(a_k - \delta_k) \geq a + \delta > 0$. This yields

$$\begin{aligned}
\prod_{i=1}^{k}(a_i - \delta_i) &\geq \left(\prod_{i=1}^{k-1} a_i - \sum_{i=1}^{k-1} \delta_i\right)(a_k - \delta_k) \\
&\geq \prod_{i=1}^{k} a_i - \sum_{i=1}^{k} \delta_i + \delta_k \left(\sum_{i=1}^{k-1} \delta_i\right) \\
&\geq \prod_{i=1}^{k} a_i - \sum_{i=1}^{k} \delta_i.
\end{aligned}$$

Indeed, the next to last inequality is equivalent to

$$\sum_{i=1}^{k} \delta_i \geq a_k \sum_{i=1}^{k-1} \delta_i + \delta_k \prod_{i=1}^{k-1} a_i,$$

which is implied by $a_k \leq 1$ and $\prod_{i=1}^{k-1} a_i \leq 1$.

We have thus shown that the inequality holds for k and this completes the induction proof. □

The connection between piercing numbers and Maximum Empty Box is highlighted by the following two lemmas.

Lemma 3. *For $a, \delta > 0$, where $a + 2\delta < 1$, and $1/\delta \in \mathbb{N}$, if $A_d(n) \geq a + 2\delta$ holds for some $n \in \mathbb{N}$, then $\tau(\mathcal{R}(\emptyset, a, \delta; d)) \geq n + 1$.*

Proof. Assume for contradiction that there exists a piercing set P with n points for $\mathcal{R}(\emptyset, a, \delta; d)$. Since $A_d(n) \geq a + 2\delta$ by assumption, there exists a hyper-rectangle r' amidst the points in P that is empty in its interior, whose volume is at least $a+2\delta$. By Lemma 1, there exists a hyper-rectangle $r \in \mathcal{R}(\emptyset, a, \delta; d)$, with $r \subset r'$ and $\mathrm{Vol}(r) = a + \delta$ and $r \cap P = \emptyset$, in contradiction to our assumption that P is a piercing set for $\mathcal{R}(\emptyset, a, \delta; d)$. This concludes the proof. □

Lemma 4. *Let $P \subset U_d$ be a finite point set and $a, \delta > 0$, where $a + 2\delta < 1$, $1/\delta \in \mathbb{N}$, and $A_d(P) \geq a + 2\delta$. Let $r'_1, \ldots, r'_j \in \mathcal{R}'(P, a, \delta; d)$ be j empty hyper-rectangles that require j piercing points in $U_d \setminus P$. Then $\tau(\mathcal{R}(P, a, \delta; d)) \geq j$.*

Proof. By Lemma 1, for every hyper-rectangle r'_i, $1 \leq i \leq j$, there exists a hyper-rectangle $r_i \in \mathcal{R}(P, a, \delta; d)$, with $r_i \subset r'_i$ and $\mathrm{Vol}(r_i) = a + \delta$ and $r_i \cap P = \emptyset$. Since piercing $r'_1, \ldots, r'_j$ requires j piercing points in $U_d \setminus P$, piercing $r_1, \ldots, r_j$ also requires j piercing points in $U_d \setminus P$. Consequently, $\tau(\mathcal{R}(P, a, \delta; d)) \geq j$. □

Piercing a set of rectangles (contained in $[0,1]^2$) whose areas are above some threshold is dual to the problem of finding a large empty rectangle (beyond this threshold) amidst the points in the piercing set. This insight could be used directly in the pursuit of a better lower bound for Wegner's inequality, it may, however, be ineffective. Here we extend the system of rectangles by adding a *grid* of segments (i.e., degenerate rectangles) as explained below. The main idea is that piercing the grid segments imposes constraints on the position of the piercing points and this allows the existence of a large empty rectangle.

2.3 Construction

All rectangles in our construction are axis-parallel and contained in the unit square $U = [0,1]^2$. Let $k \geq 2$ be a fixed integer; here we will work with $k = 2, 3, 4$. Let $a = 1/(k+1)$ and $\delta = 10^{-3}$. Note that $1/\delta \in \mathbb{N}$, as required by Lemma 1. Let $\mathcal{R}' = \mathcal{R}'(\emptyset, a, \delta; 2)$ and let $\mathcal{R} = \mathcal{R}(\emptyset, a, \delta; 2)$ be the system obtained from $\mathcal{R}'$ as in Lemma 1. Recall that $\mathcal{R}'$ is the set of all rectangles contained in U with area at least $1/(k+1) + 2\delta$; and that the area of each rectangle in (the finite family) $\mathcal{R}$ is $1/(k+1) + \delta$.

Our construction is the following finite family of rectangles (see Fig. 2 (left) for $k = 2$):

$$\mathcal{S} = \mathcal{R} \cup \mathcal{G}, \tag{15}$$

where $\mathcal{G}$ is the $k \times k$ *grid* described below.

$$\begin{aligned}
\mathcal{H} &= \{[1/(k+1), k/(k+1)] \times \{i/(k+1)\}, i = 1, 2, \ldots, k\}, \\
\mathcal{V} &= \{\{i/(k+1)\} \times [1/(k+1), k/(k+1)], i = 1, 2, \ldots, k\}, \\
\mathcal{G} &= \mathcal{H} \cup \mathcal{V}.
\end{aligned} \tag{16}$$

We show that (for every $k \geq 2$) the matching number of this family is equal to k.

Lemma 5. $\nu(\mathcal{S}) = k$.

Proof. The k (degenerate) rectangles in $\mathcal{H}$ immediately yield $\nu(\mathcal{S}) \geq k$. It remains to prove the upper bound. Let $\mathcal{I}$ be an independent set of rectangles in $\mathcal{S}$. If $\mathcal{I}$ consists only of segments in $\mathcal{G}$, it is clear that $|\mathcal{I}| \leq k$. If $\mathcal{I}$ consists only of rectangles in $\mathcal{R}$, a simple area argument implies that

$$|\mathcal{I}| \leq \left\lfloor \frac{1}{1/(k+1) + \delta} \right\rfloor \leq k.$$

Assume now that $\mathcal{I}$ consists of rectangles in $\mathcal{R}$ and grid segments in $\mathcal{G}$. Observe that any segment $s \in \mathcal{H}$ divides U into a top and a bottom region in the sense that any rectangle from $\mathcal{R}$ whose vertical extent intersects s must intersect s. A similar observation applies to segments in $\mathcal{V}$. If multiple segments are in $\mathcal{I}$, these segments must be members of the same family ($\mathcal{H}$ or $\mathcal{V}$), and the regions are further subdivided in the same manner. For each resulting region, the same area argument gives an upper bound on the number of independent rectangles from $\mathcal{R}$ in that region.

Specifically, let $h = |\mathcal{I} \cap \mathcal{H}|$; without loss of generality, we may assume that $h > 0$. Let $i_1, \ldots, i_h \in \{1, 2, \ldots, k\}$ be the subscripts of the segments in $\mathcal{I} \cap \mathcal{H}$ in ascending order. For convenience, put $i_0 = 0$ and $i_{h+1} = k + 1$. The area argument yields:

$$|\mathcal{I} \cap \mathcal{R}| \leq \sum_{j=0}^{h} \left\lfloor \frac{(i_{j+1} - i_j)/(k+1)}{1/(k+1) + \delta} \right\rfloor = \sum_{j=0}^{h} \left\lfloor \frac{i_{j+1} - i_j}{1 + (k+1)\delta} \right\rfloor$$

$$\leq \sum_{j=0}^{h} (i_{j+1} - i_j - 1) = i_{h+1} - i_0 - (h+1) = (k+1) - (h+1) = k - h.$$

Therefore, $|\mathcal{I}| = |\mathcal{I} \cap \mathcal{G}| + |\mathcal{I} \cap \mathcal{R}| \leq h + (k - h) = k$, as required. □

Key terms used in bounding the piercing number. Let P be a piercing set for $\mathcal{S}$. Let X denote the set of k^2 grid points in $\mathcal{H} \cap \mathcal{V}$. A subset of X is *independent* if no two points are on the same grid segment (that is, no two coordinates are the same). Let J denote a maximal set of independent points in $X \cap P$. Obviously $0 \leq |J| \leq k$. Assume in what follows that $|J| < k$. Let $\mathcal{H}(J)$ and $\mathcal{V}(J)$ be the grid segments pierced by J. Consider any pair of segments h, v, where $h \in \mathcal{H} \setminus \mathcal{H}(J)$ and $v \in \mathcal{V} \setminus \mathcal{V}(J)$; then h and v cannot be pierced by the common point $h \cap v$, because $J \cup \{h \cap v\}$ would be an independent set of larger cardinality than J, a contradiction. Therefore, we can view this pair of segments as "disjoint" although they share a common point. Broadly, we refer to any set of t rectangles that are pierced in $P \setminus J$ by t distinct piercing points as *quasi-disjoint* (with respect to the given J).

In our analyses we distinguish several cases depending on the size of J, as defined above, and use quasi-disjointness to infer the possible structure of a piercing set (Subsection 3.2 is the first such use).

3 Preliminary constructions

The simplest lower bound example with $\tau/\nu \geq 3/2$ is the "5-cycle" from the hypergraph setting (also mentioned in [26]): five rectangles forming a cycle where each rectangle only intersects its two neighbors in the cycle. It is worth noting that the 5-cycle can be realized with (axis-aligned) unit squares; see Fig. 1. Our construction here is more complex but the relatively simple proofs in this section pave the way for the proof of Theorem 4.

3.1 $k = 2$

According to (15) and (16), our construction consists of four segments that make $\mathcal{G}$, see Fig. 2 (left), and a finite number of rectangles with area $1/3 + \delta$.

By Lemma 5 for $k = 2$ we have $\nu(\mathcal{S}) = 2$. For example, in Fig. 2 (right), the segment $s = [1/3, 2/3] \times \{2/3\}$ divides U into the top region with area $1/3$ and the bottom region with area $2/3$. The top region can have at most $\left\lfloor \frac{1/3}{1/3+\delta} \right\rfloor = 0$ rectangles from $\mathcal{R}$ in an independent set $\mathcal{I}$, and the

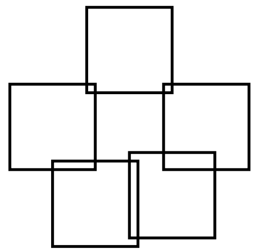

Figure 1: A 5-cycle made from unit squares.

bottom region can have at most $\left\lfloor \frac{1/3}{1/3+\delta} \right\rfloor = 1$ rectangle from $\mathcal{R}$ in $\mathcal{I}$. Thus, any independent set containing s has size at most 2; and the same bound holds for any other case.

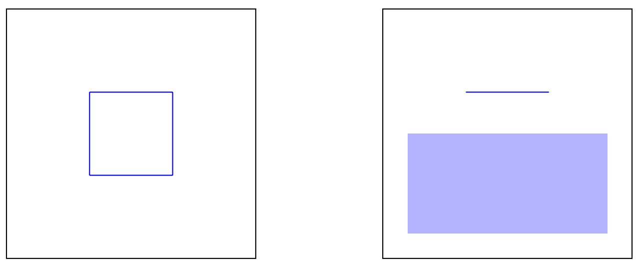

Figure 2: Left: the four segments in $\mathcal{G}$. Right: an independent set of size 2 in $\mathcal{S} = \mathcal{R} \cup \mathcal{G}$.

To see that this construction gives the ratio $\tau/\nu = 3/2$, it suffices to prove the following:

Claim 1. $\tau(\mathcal{S}) = 3$.

Proof. To see that $\tau(\mathcal{S}) \leq 3$, consider the three points shown in Fig. 3 (left), namely $(1/3, 2/3)$, $(1/2, 1/2)$, and $(2/3, 1/3)$. First note that all segments in $\mathcal{G}$ are pierced. It is easy to check that with the aforementioned three points, the maximum empty rectangle in U has area $1/3$. Therefore, all rectangles in $\mathcal{R}$ are pierced.

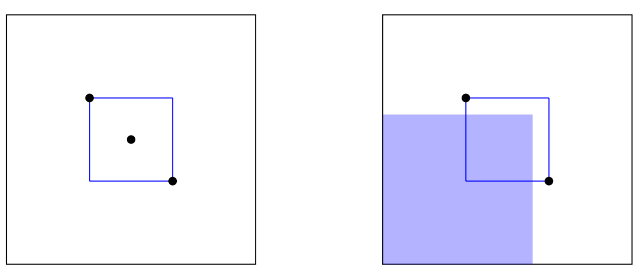

Figure 3: Left: three points piercing all rectangles in $\mathcal{S} = \mathcal{R} \cup \mathcal{G}$. Right: two points required to pierce all segments in $\mathcal{G}$ leave a rectangle from $\mathcal{R}$ unpierced.

We now prove the lower bound. Assume for contradiction that $\tau(\mathcal{S}) \leq 2$, i.e., there exist two points in U that collectively pierce the rectangles in $\mathcal{S}$. Observe that at least two points are required to pierce all segments in $\mathcal{G}$. Up to symmetry by rotation and reflection of U, there is only one case, shown in Fig. 3 (right), where the two points are $(1/3, 2/3)$ and $(2/3, 1/3)$. Consider the rectangle $r' = [0, 0.6] \times [0, 0.6] \in \mathcal{R}'$. We have $\text{Area}(r') = 0.36 > 1/3 + 2\delta$, so by Lemma 1 there is a rectangle $r \in \mathcal{R}$ that is contained in r'. Since r' is not pierced by the two points, neither is r. $\square$

3.2 $k = 3$

The first lower bound example with $\tau/\nu \geq 5/3$ was found by Wegner [33] (see also [16]) in 1968. It has 23 rectangles. A slight variation of this example was later independently found by Fon-Der-Flaass and Kostochka [17].

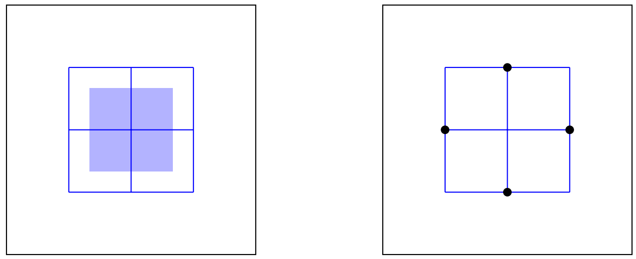

Figure 4: Left: the six segments in $\mathcal{G}$ and the center square Q. Right: $\mathcal{R} \cup \mathcal{G}$ can be pierced by 4 points.

According to (15) and (16), our construction consists of six segments making the 3×3 grid $\mathcal{G}$ and a finite number of rectangles with area $1/4 + \delta$; see Fig. 4 (left). By Lemma 5 with $k = 3$, the system $\mathcal{S} = \mathcal{G} \cap \mathcal{R}$ has $\nu = 3$; however, $\tau < 5$. In fact, $\mathcal{S}$ can be pierced by the four points $(1/4, 1/2)$, $(1/2, 1/4)$, $(1/2, 3/4)$, and $(3/4, 1/2)$, depicted in Fig. 4 (right). We therefore add the center square $Q = [1/3, 2/3] \times [1/3, 2/3]$ and redefine $\mathcal{S} := \mathcal{R} \cup \mathcal{G} \cup \{Q\}$.

Adding Q introduces a few more cases in the proof of $\nu(\mathcal{S}) = 3$, but the idea is the same as in the proof of Lemma 5 in Section 2.3; here we omit the details. To show $\tau(\mathcal{S}) \leq 5$, observe that adding the point $(1/2, 1/2)$ to the earlier set of four points is enough to pierce all rectangles in $\mathcal{S}$.

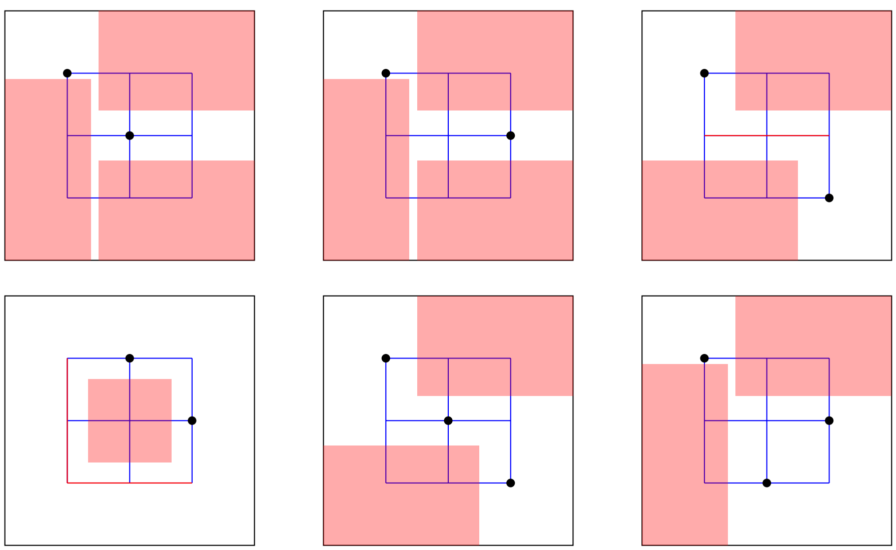

Figure 5: Four cases for $|J| = 2$ and two cases for $|J| = 3$; disjoint unpierced rectangles are shown in red. (In the 4th case for $|J| = 2$, the point $h_1 \cap v_1$ cannot be used to pierce the two red segments!)

Assume for contradiction that $\tau(\mathcal{S}) = 4$, i.e., P is a 4-point piercing set for $\mathcal{S}$. In particular, P is a piercing set for $\mathcal{G}$. Let X denote the set of 9 grid points in $\mathcal{H} \cap \mathcal{V}$. Let J denote a maximal set of independent points in $X \cap P$. Obviously $0 \leq |J| \leq 3$. Since points in J are independent, they together cover $2|J|$ grid segments. By the maximality of J, each of the remaining $|P| - |J|$ points in P can cover at most one new grid segment from the remaining $6 - 2|J|$. To cover all the segments in $\mathcal{G}$, we have $|P| - |J| \geq 6 - 2|J|$ which yields $|J| \geq 2$. Up to symmetry by rotation and reflection of U, if $|J| = 2$, there are four cases. In each of them, we exhibit three unpierced quasi-disjoint rectangles from $\mathcal{R}' \cup \mathcal{G} \cup \{Q\}$, thus by Lemma 4, at least three more points are needed; however, there are only two available points in P, a contradiction. If $|J| = 3$, there are two cases. In each of them, we exhibit two disjoint unpierced rectangles from $\mathcal{R}'$, thus by Lemma 4, at least two more points are needed, see Fig. 5; however, there is only one available point in P, a contradiction.

4 Main construction: $k = 4$

In this section we prove Theorem 4. Recall that $\mathcal{S} = \mathcal{R} \cup \mathcal{G}$ where $\mathcal{R}$ consists of a finite number of rectangles with area $1/5 + \delta$ and $\mathcal{G}$ is a 4×4 grid, see Fig. 6 (left).

$$\mathcal{H} = \{h_i = [1/5, 4/5] \times \{i/5\}, i = 1, 2, 3, 4\},$$
$$\mathcal{V} = \{v_i = \{i/5\} \times [1/5, 4/5], i = 1, 2, 3, 4\},$$
$$\mathcal{G} = \mathcal{H} \cup \mathcal{V}.$$

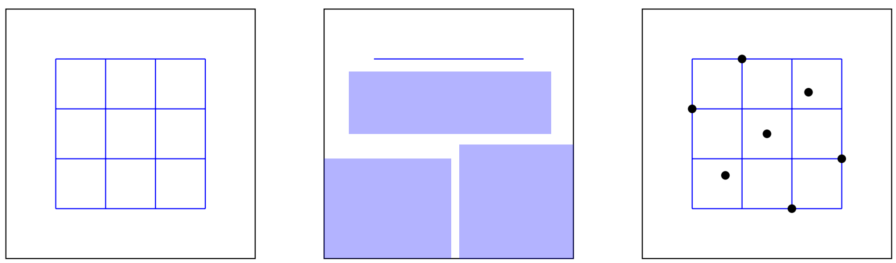

Figure 6: Left: the eight segments in $\mathcal{G}$. Middle: an independent set of size 4 in $\mathcal{S} = \mathcal{R} \cup \mathcal{G}$. Right: a set of 7 points piercing $\mathcal{S}$.

By Lemma 5 for $k = 4$ we have $\nu(\mathcal{S}) = 4$. Fig. 6 (middle) shows an independent set of size 4 in $\mathcal{S}$. To obtain Theorem 4, we need the following.

Lemma 6. $\tau(\mathcal{S}) = 7$.

Proof. The following set of 7 piercing points shows that $\tau(\mathcal{S}) \leq 7$; see Fig. 6 (right).

$$P = \left\{ \left(\frac{1}{5}, \frac{3}{5}\right), \left(\frac{1}{3}, \frac{1}{3}\right), \left(\frac{2}{5}, \frac{4}{5}\right), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{5}, \frac{1}{5}\right), \left(\frac{2}{3}, \frac{2}{3}\right), \left(\frac{4}{5}, \frac{2}{5}\right) \right\}.$$

Observe first that all the segments in $\mathcal{G}$ are pierced. It is not hard to verify that the area of the largest empty rectangle in U amidst these 7 points is equal to $1/5$. Recall that the area of every rectangle in $\mathcal{R}$ is $1/5 + \delta$, therefore all rectangles in $\mathcal{R}$ are pierced.

The proof of the lower bound $\tau(\mathcal{S}) \geq 7$ is more involved, but the idea is the same as in the earlier proofs for $k = 2$ and 3. Assume for contradiction that there exists a set P of 6 points in U that collectively pierce all the rectangles in $\mathcal{S}$. We show that piercing the grid segments in $\mathcal{G}$ imposes constraints on the position of the piercing points and this allows the existence of a large empty rectangle, i.e., one whose area is at least $1/5 + 2\delta$. By Lemma 4, this further implies the existence of an unpierced rectangle whose area is $1/5 + \delta$ in the system $\mathcal{R}$ (and thus in $\mathcal{S}$), which contradicts the assumption that P is a piercing set for $\mathcal{S}$.

Let X denote the set of 16 grid points in $\mathcal{H} \cap \mathcal{V}$. Recall that $\mathcal{H} = \{h_1, h_2, h_3, h_4\}$ and $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ (labeled in ascending order of y- and x-coordinates, respectively). Let J denote a maximal set of independent points in $X \cap P$. Obviously $0 \leq |J| \leq 4$. Since points in J are independent, they together cover $2|J|$ grid segments. By the maximality of J, each of the remaining $|P|-|J|$ points in P can cover at most one new grid segment from the remaining $8-2|J|$. To cover all the segments in $\mathcal{G}$, we have $|P|-|J| \geq 8-2|J|$ which yields $|J| \geq 2$. Henceforth, we distinguish three cases depending on the size of J.

Case $|J| = 4$. Up to symmetry by rotation and reflection of U, there are 7 configurations for these 4 points, see Fig. 7. For each configuration, we provide 3 unpierced disjoint rectangles, each of area at least $1/5 + 2\delta$. This shows that there are 3 unpierced disjoint rectangles in $\mathcal{R}$ which cannot all be pierced by the remaining $|P| - |J| = 6 - 4 = 2$ points.

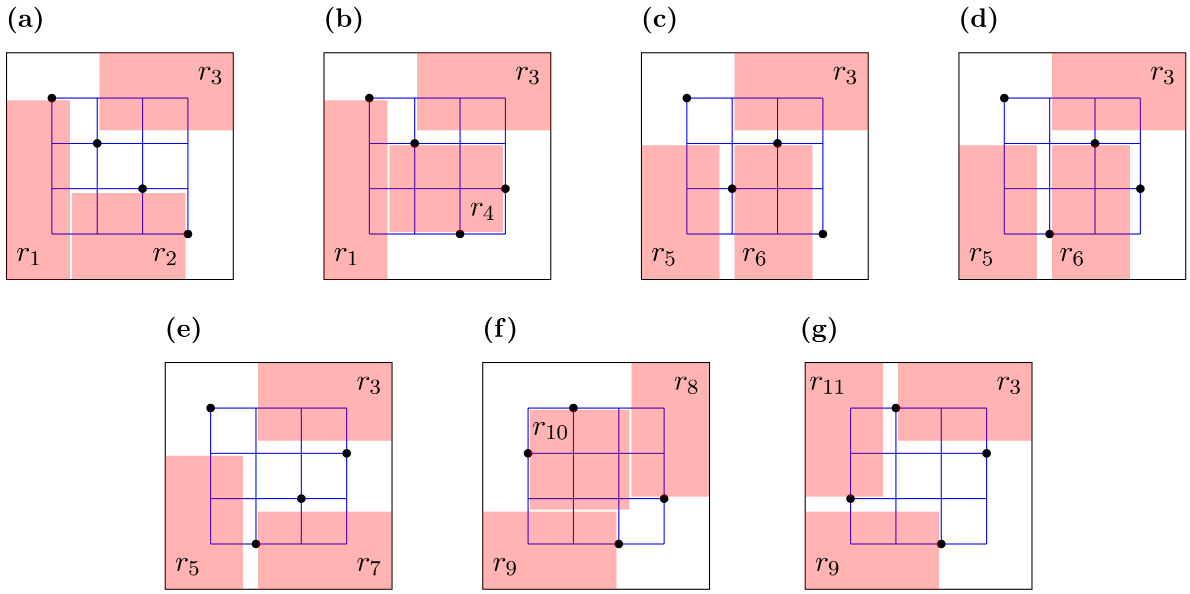

Figure 7: Seven configurations for $|J| = 4$, the unpierced disjoint rectangles are shown in red.

The following 11 rectangles, $r_1, \ldots, r_{11}$ are used. Observe that all entries in the third column of Table 3, representing *excess areas*, are nonnegative for $\delta \leq 10^{-3}$ (the bottleneck entry is r_{10}, its excess area vanishes for $\delta \approx 1/430$).

Rectangle	Dimensions	Area $-$ $(1/5 + 2\delta)$
r_1	$[0, 1/4 + 3\delta] \times [0, 4/5 - \delta]$	$3\delta/20 - 3\delta^2$
r_2	$[1/4 + 4\delta, 4/5 - \delta] \times [0, 2/5 - 2\delta]$	$1/50 - 51\delta/10 + 10\delta^2$
r_3	$[2/5 + \delta, 1] \times [2/3 - 4\delta, 1]$	$\delta/15 - 4\delta^2$
r_4	$[1/4 + 4\delta, 4/5 - \delta] \times [1/5 + \delta, 3/5 - \delta]$	$1/50 - 51\delta/10 + 10\delta^2$
r_5	$[0, 1/3 + 4\delta] \times [0, 3/5 - \delta]$	$\delta/15 - 4\delta^2$
r_6	$[2/5 + \delta, 11/15 + 5\delta] \times [0, 3/5 - \delta]$	$\delta/15 - 4\delta^2$
r_7	$[2/5 + \delta, 1] \times [0, 1/3 + 4\delta]$	$\delta/15 - 4\delta^2$
r_8	$[2/3 - 4\delta, 1] \times [2/5 + \delta, 1]$	$\delta/15 - 4\delta^2$
r_9	$[0, 3/5 - \delta] \times [0, 1/3 + 4\delta]$	$\delta/15 - 4\delta^2$
r_{10}	$[1/5 + \delta, 2/3 - 5\delta] \times [1/3 + 5\delta, 4/5 - \delta]$	$4/225 - 38\delta/5 + 36\delta^2$
r_{11}	$[0, 1/3 + 4\delta] \times [2/5 + \delta, 1]$	$\delta/15 - 4\delta^2$

Table 3: List of rectangles used for the case $|J| = 4$.

The argument is summarized in the following table:

Configuration	J	3 unpierced disjoint rectangles
(a)	$\{(1/5,4/5),(2/5,3/5),(3/5,2/5),(4/5,1/5)\}$	r_1, r_2, r_3
(b)	$\{(1/5,4/5),(2/5,3/5),(3/5,1/5),(4/5,2/5)\}$	r_1, r_3, r_4
(c)	$\{(1/5,4/5),(2/5,2/5),(3/5,3/5),(4/5,1/5)\}$	r_3, r_5, r_6
(d)	$\{(1/5,4/5),(2/5,1/5),(3/5,3/5),(4/5,2/5)\}$	r_3, r_5, r_6
(e)	$\{(1/5,4/5),(2/5,1/5),(3/5,2/5),(4/5,3/5)\}$	r_3, r_5, r_7
(f)	$\{(1/5,3/5),(2/5,4/5),(3/5,1/5),(4/5,2/5)\}$	r_8, r_9, r_{10}
(g)	$\{(1/5,2/5),(2/5,4/5),(3/5,1/5),(4/5,3/5)\}$	r_3, r_9, r_{11}

Table 4: $|J| = 4$. All 7 configurations are handled by providing three unpierced disjoint rectangles.

Case $|J| = 3$. Up to symmetry by rotation and reflection of U, there are 16 configurations for these 3 points, see Fig. 8. For each configuration, there are $|\mathcal{G}| - 2|J| = 8 - 6 = 2$ unpierced grid segments. These two segments must intersect otherwise J is not independent. However they cannot be pierced by one point because otherwise the configuration would have been handled in the previous case where $|J| = 4$; as such, they are quasi-disjoint. Out of the 16 configurations, we handle 13 using the same technique as in the previous case. That is, we exhibit 4 unpierced quasi-disjoint rectangles, each of them either is a grid segment (i.e., in $\mathcal{G}$) or has area at least $1/5 + 2\delta$ (i.e., in $\mathcal{R}'$). This shows that there are 4 unpierced quasi-disjoint rectangles in $\mathcal{G} \cup \mathcal{R}$ which cannot all be pierced by the remaining $|P| - |J| = 6 - 3 = 3$ points. The other three configurations are handled differently.

The following 6 additional rectangles are used. Observe that all entries in the third column of Table 5, representing excess areas, are nonnegative for $\delta \leq 10^{-3}$ (the bottleneck entries are r_{12}, r_{15}, r_{16}, and r_{17}, their excess areas vanish for $\delta = 1/60$).

Rectangle	Dimensions	Area $-$ $(1/5 + 2\delta)$
r_{12}	$[0, 3/5 - \delta] \times [1/5 + \delta, 8/15 + 5\delta]$	$\delta/15 - 4\delta^2$
r_{13}	$[1/5 + \delta, 4/5 - \delta] \times [2/3 - 5\delta, 1]$	$\delta/3 - 10\delta^2$
r_{14}	$[0, 1/3 + 5\delta] \times [1/5 + \delta, 4/5 - \delta]$	$\delta/3 - 10\delta^2$
r_{15}	$[0, 3/5 - \delta] \times [2/5 + \delta, 11/15 + 5\delta]$	$\delta/15 - 4\delta^2$
r_{16}	$[2/3 - 4\delta, 1] \times [0, 3/5 - \delta]$	$\delta/15 - 4\delta^2$
r_{17}	$[7/15 - 5\delta, 4/5 - \delta] \times [2/5 + \delta, 1]$	$\delta/15 - 4\delta^2$

Table 5: List of additional rectangles used for the case $|J| = 3$.

Arguments for the first 13 configurations are summarized in the following table:

Configuration	J	4 unpierced quasi-disjoint rectangles
(a)	$\{(1/5,4/5),(2/5,3/5),(3/5,2/5)\}$	h_1, v_4, r_{12}, r_{13}
(b)	$\{(1/5,4/5),(2/5,3/5),(4/5,2/5)\}$	h_1, v_3, r_8, r_{12}
(c)	$\{(1/5,4/5),(2/5,3/5),(4/5,1/5)\}$	h_2, v_3, r_8, r_9
(d)	$\{(1/5,4/5),(3/5,3/5),(4/5,2/5)\}$	h_1, v_2, r_3, r_{14}

(e)	$\{(1/5, 4/5), (2/5, 1/5), (3/5, 3/5)\}$	h_2, r_7, r_8, r_{15}
(f)	$\{(1/5, 4/5), (3/5, 3/5), (4/5, 1/5)\}$	v_2, r_3, r_5, r_6
(g)	$\{(1/5, 4/5), (3/5, 2/5), (4/5, 3/5)\}$	v_2, r_3, r_5, r_{16}
(h)	$\{(1/5, 4/5), (3/5, 1/5), (4/5, 3/5)\}$	v_2, r_3, r_5, r_{16}
(i)	$\{(1/5, 4/5), (3/5, 1/5), (4/5, 2/5)\}$	h_3, v_2, r_3, r_5
(j)	$\{(1/5, 3/5), (2/5, 4/5), (4/5, 2/5)\}$	h_1, v_3, r_8, r_{12}
(k)	$\{(1/5, 2/5), (2/5, 4/5), (3/5, 3/5)\}$	r_3, r_9, r_{11}, r_{16}
(l)	$\{(2/5, 4/5), (3/5, 3/5), (4/5, 2/5)\}$	h_1, v_1, r_3, r_4
(m)	$\{(1/5, 2/5), (2/5, 4/5), (4/5, 3/5)\}$	v_3, r_9, r_{11}, r_{16}

Table 6: $|J| = 3$. The first 13 configurations are handled by exhibiting 4 unpierced quasi-disjoint rectangles.

For each of the remaining three configurations, we exhibit 7 unpierced rectangles, each of which is either a grid segment (i.e., in $\mathcal{G}$) or has area at least $1/5+2\delta$ (i.e., in $\mathcal{R}'$). Observe that each point in $U \setminus X$ (recall that X is the set of grid points) can cover at most 2 of these 7 rectangles. Therefore, at least 4 more piercing points are needed. The arguments are summarized in the following table:

Configuration	J	7 unpierced rectangles
(n)	$\{(1/5, 4/5), (2/5, 2/5), (3/5, 3/5)\}$	$h_1, v_4, r_3, r_9, r_{14}, r_{15}, r_{16}$
(o)	$\{(1/5, 4/5), (2/5, 1/5), (4/5, 3/5)\}$	$h_2, v_3, r_3, r_5, r_7, r_{15}, r_{16}$
(p)	$\{(1/5, 3/5), (2/5, 4/5), (3/5, 2/5)\}$	$h_1, v_4, r_3, r_5, r_7, r_{12}, r_{17}$

Table 7: $|J| = 3$. The remaining 3 configurations are handled by exhibiting 7 unpierced rectangles that need at least 4 additional piercing points.

Case $|J| = 2$. Up to symmetry by rotation and reflection of U, there are 13 configurations for these 2 points, see Fig. 9. For each configuration, there are $|G| - 2|J| = 8 - 4 = 4$ unpierced grid segments. As in the previous case, these segments are quasi-disjoint (i.e., cannot be pierced at their intersections) because otherwise the configuration would have been handled in the previous cases where $|J| = 3$ or 4. Note that there remain $|P| - |J| = 6 - 2 = 4$ points, so they must all be placed on segments in $\mathcal{G}$, one for each unpierced segment.

Out of the 13 configurations, we handle the first 8 using the same technique as in the previous cases. That is, we provide 5 unpierced quasi-disjoint rectangles, each of them either is a grid segment (i.e., in $\mathcal{G}$) or has area at least $1/5 + 2\delta$ (i.e., in $\mathcal{R}'$). This shows that there are 5 unpierced quasi-disjoint rectangles in $\mathcal{G} \cup \mathcal{R}$ which cannot all be pierced by the remaining 4 points. The remaining 5 configurations are handled differently.

The following 13 additional rectangles are used. Observe that all entries in the third column of Table 8 are nonnegative for $\delta \le 10^{-3}$ (the bottleneck entries are $r_{20}, r_{21}, r_{22}, r_{23}, r_{27}$, and r_{30}, their excess areas vanish for $\delta \approx 1/255$).

Rectangle	Dimension	Area $-$ $(1/5 + 2\delta)$
r_{18}	$[2/5 + \delta, 1] \times [1/5 + \delta, 8/15 + 5\delta]$	$\delta/15 - 4\delta^2$
r_{19}	$[1/5 + \delta, 8/15 + 5\delta] \times [0, 3/5 - \delta]$	$\delta/15 - 4\delta^2$
r_{20}	$[0, 11/20 - 5\delta] \times [0, 2/5 - 2\delta]$	$1/50 - 51\delta/10 + 10\delta^2$
r_{21}	$[0, 11/20 - 5\delta] \times [1/5 + \delta, 3/5 - \delta]$	$1/50 - 51\delta/10 + 10\delta^2$

r_{22}	$[2/5+\delta, 4/5-\delta] \times [9/20+5\delta, 1]$	$1/50 - 51\delta/10 + 10\delta^2$
r_{23}	$[3/5+2\delta, 1] \times [9/20+5\delta, 1]$	$1/50 - 51\delta/10 + 10\delta^2$
r_{24}	$[11/20-3\delta, 1] \times [0, 9/20+3\delta]$	$1/400 + 7\delta/10 + 9\delta^2$
r_{25}	$[1/5+\delta, 8/15+6\delta] \times [1/5+\delta, 4/5-\delta]$	$\delta/3 - 10\delta^2$
r_{26}	$[0, 9/20+3\delta, 1] \times [0, 9/20+3\delta]$	$1/400 + 7\delta/10 + 9\delta^2$
r_{27}	$[0, 2/5-2\delta] \times [9/20+5\delta, 1]$	$1/50 - 51\delta/10 + 10\delta^2$
r_{28}	$[7/15-5\delta, 4/5-\delta] \times [2/5+\delta, 1]$	$\delta/15 - 4\delta^2$
r_{29}	$[2/5+\delta, 1] \times [7/15-5\delta, 4/5-\delta]$	$\delta/15 - 4\delta^2$
r_{30}	$[9/20+5\delta, 1] \times [0, 2/5-2\delta]$	$1/50 - 51\delta/10 + 10\delta^2$

Table 8: List of additional rectangles used for the case $|J| = 2$.

Arguments for the first 8 configurations are summarized in the following table:

Configuration	J	5 unpierced quasi-disjoint rectangles
(a)	$\{(1/5, 4/5), (3/5, 3/5)\}$	$h_1, v_2, r_3, r_{14}, r_{18}$
(b)	$\{(1/5, 4/5), (4/5, 3/5)\}$	$h_1, v_2, r_3, r_{14}, r_{18}$
(c)	$\{(1/5, 4/5), (3/5, 2/5)\}$	h_3, v_2, r_3, r_5, r_7
(d)	$\{(1/5, 4/5), (4/5, 2/5)\}$	h_3, v_2, r_3, r_5, r_7
(e)	$\{(1/5, 4/5), (4/5, 2/5)\}$	$h_2, v_3, r_8, r_9, r_{15}$
(f)	$\{(2/5, 4/5), (3/5, 2/5)\}$	$h_3, v_1, r_3, r_{16}, r_{19}$
(g)	$\{(2/5, 4/5), (4/5, 2/5)\}$	h_1, h_3, v_1, r_3, r_4
(h)	$\{(2/5, 4/5), (3/5, 1/5)\}$	$h_3, v_1, r_3, r_{16}, r_{19}$

Table 9: $|J| = 2$. The first 8 configurations are handled by providing 5 unpierced quasi-disjoint rectangles.

There are 5 more configurations. For each of them, we exhibit 9 unpierced rectangles, each of which is either a grid segment (i.e., in $\mathcal{G}$) or has area at least $1/5 + 2\delta$ (i.e., in $\mathcal{R}'$). Recall that if $|P| = 6$, the remaining 4 piercing points must all lie on the grid segments $\mathcal{G} \setminus X$ (X is the set of grid points). But each point in $\mathcal{G} \setminus X$ can cover at most 2 of these 9 rectangles, which is a contradiction. The arguments are summarized in the following table:

Configuration	J	9 unpierced rectangles
(i)	$\{(1/5, 4/5), (2/5, 3/5)\}$	$h_1, h_2, v_3, v_4, r_{20}, r_{21}, r_{22}, r_{23}, r_{24}$
(j)	$\{(1/5, 3/5), (2/5, 4/5)\}$	$h_1, h_2, v_3, v_4, r_{20}, r_{21}, r_{22}, r_{23}, r_{24}$
(k)	$\{(2/5, 4/5), (3/5, 3/5)\}$	$h_1, h_2, v_1, v_4, r_3, r_9, r_{11}, r_{16}, r_{25}$
(l)	$\{(2/5, 4/5), (4/5, 3/5)\}$	$h_1, h_2, v_1, v_3, r_3, r_9, r_{11}, r_{16}, r_{25}$
(m)	$\{(2/5, 3/5), (3/5, 2/5)\}$	$h_1, h_4, v_1, v_4, r_{26}, r_{27}, r_{28}, r_{29}, r_{30}$

Table 10: $|J| = 2$. The remaining 5 configurations are handled by providing 9 unpierced rectangles that cannot be pierced by 4 points in $\mathcal{G} \setminus X$.

This concludes the proof of Lemma 6. □

Now Theorem 4 immediately follows from Lemma 5 (with $k = 4$) and Lemma 6.

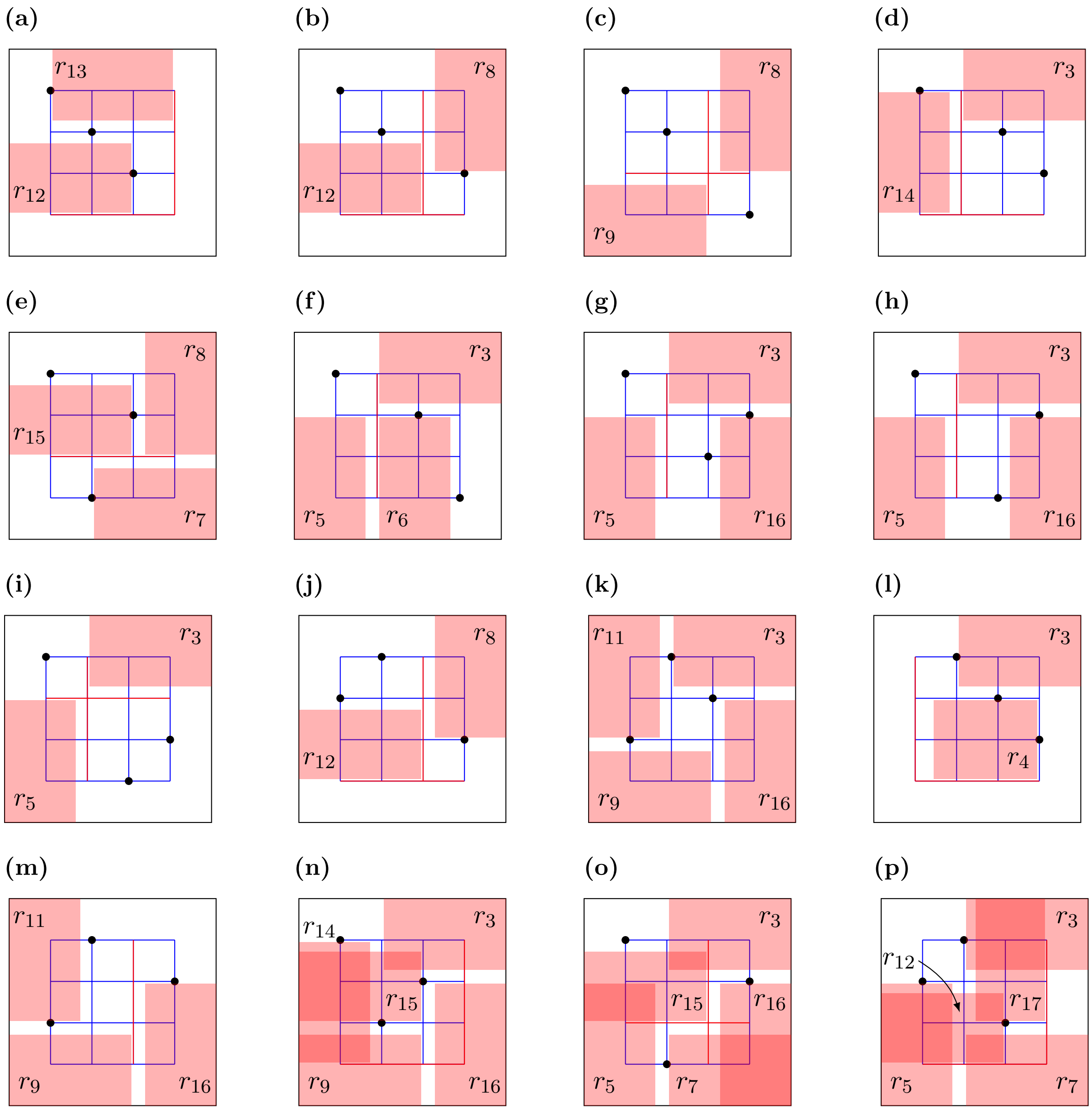

Figure 8: 16 configurations for $|J| = 3$. For the first 13 configurations, the unpierced quasi-disjoint rectangles are shown in red; for the last 3 configurations, the 7 unpierced rectangles that need at least 4 additional piercing points are shown in red.

Remarks.

1. An alternative proof of Theorem 4 could be obtained by restricting rectangles in $\mathcal{R}$ to those that are used in the proof of the lower bound on τ (Lemma 6) and their images under rotation and reflection of U. This would make the resulting lower bound example smaller with regard to number of rectangles in it. But its description would be tedious and not as enlightening as our presentation here.

2. A natural question is whether our construction can be used to create lower bound examples for larger k. Interestingly enough, for $k = 5$ the rectangles in $\mathcal{S}$ can be pierced by 8 points. In Fig. 10, all the segments in $\mathcal{G}$ are pierced and the area of the largest empty rectangle is 1/6

Figure 9: 13 configurations for $|J| = 2$. For the first 8 configurations, the unpierced quasi-disjoint rectangles are shown in red; for the last 5 configurations, the 9 unpierced rectangles that cannot be pierced by 4 points in $\mathcal{G} \setminus X$ are shown in red.

so all rectangles in $\mathcal{R}$ are also pierced. Recall that in the construction for $k = 3$, we added a center square to increase τ by 1. A similar fix might be possible (but in view of Fig. 10, a center square clearly won't work). Another major difficulty is that the number of cases required to prove a lower bound on τ grows rapidly with respect to k.

3. The straightforward method to construct lower bounds by taking unions yields the following:

$$f(m+n) \geq f(m) + f(n), \text{ for every } m, n \geq 0. \tag{17}$$

By (17) and Theorem 4 we have $f(5) \geq f(4) + f(1) \geq 7 + 1 = 8$; alternatively, the result follows from $f(5) \geq f(3) + f(2) = 5 + 3 = 8$.

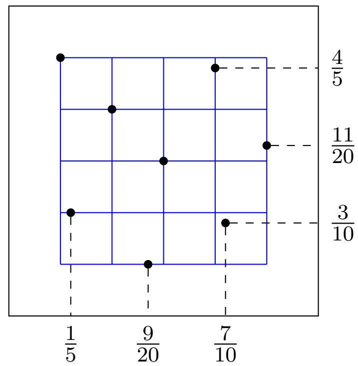

Figure 10: The system $\mathcal{S} = \mathcal{G} \cup \mathcal{R}$ for $k = 5$ can be pierced by 8 points.

5 Higher dimensions

Proof of Theorem 5. By (11), we have $c_d \geq \frac{\log d}{4} \geq \frac{10}{4}$. By the definition of c_d, see (10), there exist arbitrarily large integers n such that $nA_d(n) \geq c_d - 1$. Let $k = \lfloor \frac{n}{c_d-1} \rfloor + 1$. Since n can be arbitrarily large, we may assume that $k \geq 10$. On one hand, we have $k > \frac{n}{c_d-1}$ and thus $n < (c_d - 1)k$. On the other hand, we have $k \leq \frac{n}{c_d-1} + 1$, and therefore $n \geq (k-1)(c_d - 1)$. Note that $c_d \geq 10/4$ and $k \geq 10$ imply that $kc_d/2 \geq k + c_d$.

Consider the system of boxes

$$\mathcal{R}' := \mathcal{R}'\left(\emptyset, \frac{1}{k+1}, \frac{1}{4k^2}; d\right).$$

Observe that the preconditions for Lemma 1 are trivially met; and so the system

$$\mathcal{R} := \mathcal{R}\left(\emptyset, \frac{1}{k+1}, \frac{1}{4k^2}; d\right)$$

exists. Note that $\nu(\mathcal{R}) = k$. Since

$$A_d(n) \geq \frac{c_d - 1}{n} > \frac{1}{k} > \frac{1}{k+1} + \frac{2}{4k^2} \quad \text{(for } k \geq 10\text{)},$$

Lemma 3 yields

$$\begin{aligned}\tau(\mathcal{R}) &= \tau\left(\mathcal{R}\left(\emptyset, \frac{1}{k+1}, \frac{1}{4k^2}; d\right)\right)\\ &\geq n + 1 \geq (k-1)(c_d - 1) + 1\\ &= kc_d - (k + c_d) + 2 > \frac{kc_d}{2}.\end{aligned}$$

Consequently, we have $\tau(\mathcal{R}) \geq \nu(\mathcal{R})c_d/2$, or $\tau(\mathcal{R})/\nu(\mathcal{R}) \geq c_d/2$, as required. □

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