

# New bounds on the average distance from the Fermat-Weber center of a planar convex body

Adrian Dumitrescu\*      Minghui Jiang†      Csaba D. Tóth‡

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## Abstract

The Fermat-Weber center of a planar body  $Q$  is a point in the plane from which the average distance to the points in  $Q$  is minimal. We first show that for any convex body  $Q$  in the plane, the average distance from the Fermat-Weber center of  $Q$  to the points in  $Q$  is larger than  $\frac{1}{6} \cdot \Delta(Q)$ , where  $\Delta(Q)$  is the diameter of  $Q$ . This proves a conjecture of Carmi, Har-Peled and Katz. From the other direction, we prove that the same average distance is at most  $\frac{2(4-\sqrt{3})}{3\sqrt{3}} \cdot \Delta(Q) < 0.3490 \cdot \Delta(Q)$ . The new bound substantially improves the previous bound of  $\frac{2^{13}}{3\sqrt{3}} \cdot \Delta(Q) \approx 0.3849 \cdot \Delta(Q)$  due to Abu-Affash and Katz, and brings us closer to the conjectured value of  $\frac{1}{3} \cdot \Delta(Q)$ . We also confirm the upper bound conjecture for centrally symmetric planar convex bodies.

## 1 Introduction

The Fermat-Weber center of a measurable planar set  $Q$  with positive area is a point in the plane that minimizes the average distance to the points in  $Q$ . Such a point is the ideal location for a base station (e.g., fire station or a supply station) serving the region  $Q$ , assuming the region has uniform density. Given a measurable set  $Q$  with positive area and a point  $p$  in the plane, let  $\mu_Q(p)$  be the average distance between  $p$  and the points in  $Q$ , namely,

$$\mu_Q(p) = \frac{\int_{q \in Q} \text{dist}(p, q) \, dq}{\text{area}(Q)},$$

where  $\text{dist}(p, q) = |pq|$  is the Euclidean distance between  $p$  and  $q$ . Let  $FW_Q$  be the Fermat-Weber center of  $Q$ , and write  $\mu_Q^* = \min\{\mu_Q(p) : p \in \mathbb{R}^2\} = \mu_Q(FW_Q)$ .

Carmi, Har-Peled and Katz [3] showed that there exists a constant  $c > 0$  such that  $\mu_Q^* \geq c \cdot \Delta(Q)$  holds for any convex body  $Q$ , where  $\Delta(Q)$  denotes the diameter of  $Q$ . The convexity is necessary, since it is easy to construct nonconvex regions where the average distance from the Fermat-Weber center is arbitrarily small compared to the diameter. Of course the opposite inequality  $\mu_Q^* \leq c' \cdot \Delta(Q)$  holds for any body  $Q$  (convexity is not required), since we can trivially take  $c' = 1$ .

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\*Department of Computer Science, University of Wisconsin–Milwaukee, WI 53201-0784, USA. Email: [ad@cs.uwm.edu](mailto:ad@cs.uwm.edu). Supported in part by NSF CAREER grant CCF-0444188.

†Department of Computer Science, Utah State University, Logan, UT 84322-4205, USA. Email: [mjiang@cc.usu.edu](mailto:mjiang@cc.usu.edu). Supported in part by NSF grant DBI-0743670.

‡Department of Mathematics, University of Calgary, AB, Canada T2N 1N4. E-mail: [cdtoth@ucalgary.ca](mailto:cdtoth@ucalgary.ca). Supported in part by NSERC grant RGPIN 35586.

Let  $c_1$  denote the infimum, and  $c_2$  denote the supremum of  $\mu_Q^*/\Delta(Q)$  over all convex bodies  $Q$  in the plane. Carmi, Har-Peled and Katz [3] conjectured that  $c_1 = \frac{1}{6}$  and  $c_2 = \frac{1}{3}$ . Moreover, they conjectured that the supremum  $c_2$  is attained for a circular disk  $D$ , where  $\mu_D^* = \frac{1}{3} \cdot \Delta(D)$ . They also proved that  $\frac{1}{7} \leq c_1 \leq \frac{1}{6}$ . The inequality  $c_1 \leq \frac{1}{6}$  is given by an infinite sequence of rhombi,  $P_\varepsilon$ , where one diagonal has some fixed length, say 2, and the other diagonal tends to zero; see Fig. 1. By symmetry, the Fermat-Weber center of a rhombus is its center of symmetry, and one can verify that  $\mu_{P_\varepsilon}^*/\Delta(P_\varepsilon)$  tends to  $\frac{1}{6}$ . The lower bound for  $c_1$  has been recently further improved by Abu-Affash and Katz [1] from  $\frac{1}{7}$  to  $\frac{4}{25}$ . Here we establish that  $c_1 = \frac{1}{6}$  and thereby confirm the first of the two conjectures of Carmi, Har-Peled and Katz.

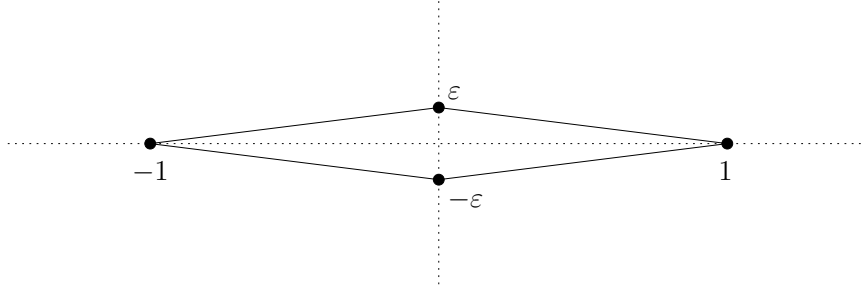


Figure 1: A flat rhombus  $P_\varepsilon$ , with  $\lim_{\varepsilon \rightarrow 0} \mu_{P_\varepsilon}^*/\Delta(P_\varepsilon) = \frac{1}{6}$ .

Regarding the second conjecture, recently Abu-Affash and Katz proved that  $c_2 \leq \frac{2}{3\sqrt{3}} = 0.3849\dots$ . Here we further improve this bound and bring it closer to the conjectured value of  $\frac{1}{3}$ . Finally, we also confirm the upper bound conjecture for centrally symmetric convex bodies  $Q$ .

Our main results are summarized in the following two theorems:

**Theorem 1.** *For any convex body  $Q$  in the plane, we have  $\mu_Q^* > \frac{1}{6} \cdot \Delta(Q)$ .*

**Theorem 2.** *For any convex body  $Q$  in the plane, we have*

$$\mu_Q^* \leq \frac{2(4 - \sqrt{3})}{13} \cdot \Delta(Q) < 0.3490 \cdot \Delta(Q).$$

*Moreover, if  $Q$  is centrally symmetric, then  $\mu_Q^* \leq \frac{1}{3} \cdot \Delta(Q)$ .*

**Remark 1.** The average distance from a point  $p$  in the plane can be defined analogously for curves or for finite point sets. Observe that for a line segment  $I$  (a one-dimensional convex set, but also a curve), we have  $\mu_I^*/\Delta(I) = \frac{1}{4}$ . It may be interesting to note that while the thin rhombi mentioned above tend in the limit to a line segment, the value of the limit  $\mu_{P_\varepsilon}^*/\Delta(P_\varepsilon)$  equals  $\frac{1}{6}$ , not  $\frac{1}{4}$ . An easy construction shows that for finite point sets  $Q$ , even in convex position, there is no positive constant  $c$  such that  $\mu_Q^* \geq c \cdot \Delta(Q)$ . For instance take  $n - 1$  points close to  $(0, 0)$  and one point at  $(1, 0)$ ; then  $\mu_Q^* \approx 2/n \rightarrow 0$ , while  $\Delta(Q) = 1$ .

**Remark 2.** In some applications, the cost of serving a location  $q$  from a facility at point  $p$  is  $\text{dist}^\kappa(p, q)$  for some exponent  $\kappa \geq 1$ , rather than  $\text{dist}(p, q)$ . We can define

$$\mu_Q^\kappa(p) = \left( \int_{q \in Q} \text{dist}^\kappa(p, q) \, dq \right) / \text{area}(Q) \quad \text{and} \quad \mu_Q^{\kappa*} = \inf \{ \mu_Q^\kappa(p) : p \in \mathbb{R}^2 \},$$

which is invariant under congruence. The ratio  $\mu_Q^{\kappa^*}/\Delta^\kappa(Q)$  is also invariant under similarity. The proof of Theorem 1 carries over for this variant and shows that  $\mu_Q^{\kappa^*}/\Delta^\kappa(Q) > \frac{1}{(\kappa+2)2^\kappa}$  for any convex body  $Q$ , and  $\lim_{\varepsilon \rightarrow 0} \mu_{P_\varepsilon}^{\kappa^*}/2^\kappa = \frac{1}{(\kappa+2)2^\kappa}$ .

For the upper bound, the picture is not so clear:  $\mu_Q^{\kappa^*}/\Delta(Q)$  is conjectured to be maximal for the circular disk, however, there is a  $\kappa \geq 1$  such that  $\mu_Q^{\kappa^*}/\Delta^\kappa(Q)$  cannot be maximal for the disk. In particular, if  $D$  is a disk of diameter 2 and  $R$  is a convex body of diameter 2 whose smallest enclosing disk has diameter more than 2 (e.g., a regular or a Reuleaux triangle of diameter 2), then  $\mu_D^{\kappa^*} < \mu_R^{\kappa^*}$ , for a sufficiently large  $\kappa > 1$ . Let  $o$  be an arbitrary point in the plane, and let  $D$  be centered at  $o$ . Then  $\int_{q \in D} \text{dist}^\kappa(o, q) \, dq = \int_0^{2\pi} \int_0^1 r^\kappa \cdot r \, dr \, d\theta = \frac{2\pi}{\kappa+2}$ , and so  $\lim_{\kappa \rightarrow \infty} \mu_D^{\kappa^*} \leq \lim_{\kappa \rightarrow \infty} \frac{2}{\kappa+2} = 0$ . On the other hand, for any region  $R'$  lying outside of  $D$  and for any  $\kappa \geq 1$ , we have  $\int_{q \in R'} \text{dist}^\kappa(o, q) \, dq \geq \text{area}(R') > 0$ . If  $R' = R \setminus D$  is the part of  $R$  lying outside  $D$ , then  $\lim_{\kappa \rightarrow \infty} \mu_R^{\kappa^*} \geq \text{area}(R')/\pi > 0$ .

**Related work.** Fekete, Mitchell, and Weinbrecht [8] studied a continuous version of the problem for polygons with holes, where the distance between two points is measured by the  $L_1$  geodesic distance. A related question on Fermat-Weber centers in a discrete setting deals with stars and Steiner stars [5, 7]. The reader can find more information on other variants of the Fermat-Weber problem in [4, 11].

## 2 Lower bound: proof of Theorem 1

In a nutshell the proof goes as follows. Given a convex body  $Q$ , we take its Steiner symmetrization with respect to a supporting line of a diameter segment  $cd$ , followed by another Steiner symmetrization with respect to the perpendicular bisector of  $cd$ . The two Steiner symmetrizations preserve the area and the diameter, and do not increase the average distance from the corresponding Fermat-Weber centers. In the final step, we prove that the inequality holds for a convex body with two orthogonal symmetry axes.

**Steiner symmetrization with respect to an axis.** Steiner symmetrization of a convex figure  $Q$  with respect to an axis (line)  $\ell$  consists in replacing  $Q$  by a new figure  $S(Q, \ell)$  with symmetry axis  $\ell$  by means of the following construction: Each chord of  $Q$  orthogonal to  $\ell$  is displaced along its line to a new position where it is symmetric with respect to  $\ell$ , see [12, pp. 64]. The resulting figure  $S(Q, \ell)$  is also convex, and obviously has the same area as  $Q$ .

A body  $Q$  is  $x$ -monotone if the intersection of  $Q$  with every vertical line is either empty or is connected (that is, a point or a line segment). Every  $x$ -monotone body  $Q$  is bounded by the graphs of some functions  $f : [a, b] \rightarrow \mathbb{R}$  and  $g : [a, b] \rightarrow \mathbb{R}$  such that  $g(x) \leq f(x)$  for all  $x \in [a, b]$ . The Steiner symmetrization with respect to the  $x$ -axis  $\ell_x$  transforms  $Q$  into an  $x$ -monotone body  $S(Q, \ell_x)$  bounded by the functions  $\frac{1}{2}(f(x) - g(x))$  and  $\frac{1}{2}(g(x) - f(x))$  for  $x \in [a, b]$ . As noted earlier,  $\text{area}(S(Q, \ell_x)) = \text{area}(Q)$ . The next two lemmas do not require the convexity of  $Q$ .

**Lemma 1.** *Let  $Q$  be an  $x$ -monotone body in the plane with a diameter parallel or orthogonal to the  $x$ -axis, then  $\Delta(Q) = \Delta(S(Q, \ell_x))$ .*

**Proof.** Let  $Q' = S(Q, \ell_x)$ . If  $Q$  has a diameter parallel to the  $x$ -axis, then the diameter is  $[(a, c), (b, c)]$ , with a value  $c \in \mathbb{R}$ ,  $g(a) = c = f(a)$  and  $g(b) = c = f(b)$ . That is,  $\Delta(Q) = b - a$ . In this case, the diameter of  $Q'$  is at least  $b - a$ , since both points  $(a, 0)$  and  $(b, 0)$  are in  $Q'$ . If  $Q$  has a diameter orthogonal to the  $x$ -axis, then the diameter is  $[(x_0, f(x_0)), (x_0, g(x_0))]$  for some

$x_0 \in [a, b]$ , and  $\Delta(Q) = f(x_0) - g(x_0)$ . In this case, the diameter of  $Q'$  is at least  $f(x_0) - g(x_0)$ , since both points  $(x_0, \frac{1}{2}(f(x_0) - g(x_0)))$  and  $(x_0, \frac{1}{2}(g(x_0) - f(x_0)))$  are in  $Q'$ . Therefore, we have  $\Delta(Q') \geq \Delta(Q)$ .

Let  $A_1$  and  $A_2$  be two points on the boundary of  $Q'$  such that  $\Delta(Q') = \text{dist}(A_1, A_2)$ . Since  $Q'$  is symmetric to the  $x$ -axis, points  $A_1$  and  $A_2$  cannot both be on the upper (resp., lower) boundary of  $Q'$ . Assume w.l.o.g. that  $A_1 = (x_1, \frac{1}{2}(f(x_1) - g(x_1)))$  and  $A_2 = (x_2, \frac{1}{2}(g(x_2) - f(x_2)))$  for some  $a \leq x_1, x_2 \leq b$ .

$$\Delta(Q') = \text{dist}(A_1, A_2) = \sqrt{(x_2 - x_1)^2 + \left(\frac{f(x_1) + f(x_2) - g(x_1) - g(x_2)}{2}\right)^2}.$$

Now consider the following two point pairs in  $Q$ . The distance between  $B_1 = (x_1, f(x_1))$  and  $B_2 = (x_2, g(x_2))$  is  $\text{dist}(B_1, B_2) = \sqrt{(x_2 - x_1)^2 + (f(x_1) - g(x_2))^2}$ . Similarly, the distance between  $C_1 = (x_1, g(x_1))$  and  $C_2 = (x_2, f(x_2))$  is  $\text{dist}(C_1, C_2) = \sqrt{(x_2 - x_1)^2 + (g(x_1) - f(x_2))^2}$ . Using the inequality between the arithmetic and quadratic means, we have

$$\left(\frac{f(x_1) + f(x_2) - g(x_1) - g(x_2)}{2}\right)^2 \leq \frac{(f(x_1) - g(x_2))^2 + (g(x_1) - f(x_2))^2}{2}.$$

This implies that  $\text{dist}(A_1, A_2) \leq \max(\text{dist}(B_1, B_2), \text{dist}(C_1, C_2))$ , and so  $\Delta(Q') \leq \Delta(Q)$ . We conclude that  $\Delta(Q) = \Delta(S(Q, \ell_x))$ .  $\square$

**Lemma 2.** *If  $Q$  is an  $x$ -monotone body in the plane, then  $\mu_Q^* \geq \mu_{S(Q, \ell_x)}^*$ .*

**Proof.** If  $(x_0, y_0)$  is the Fermat-Weber center of  $Q$ , then

$$\mu_Q^* = \frac{\int_a^b \int_{g(x)}^{f(x)} \sqrt{(x - x_0)^2 + (y - y_0)^2} dy dx}{\text{area}(Q)}.$$

Observe that  $\int_{g(x)}^{f(x)} \sqrt{(x - x_0)^2 + (y - y_0)^2} dy$  is the integral of the distances of the points in a line segment of length  $f(x) - g(x)$  from a point at distance  $|x - x_0|$  from the supporting line of the segment. This integral is minimal if the point is on the orthogonal bisector of the segment. That is, we have

$$\begin{aligned} \int_{g(x)}^{f(x)} \sqrt{(x - x_0)^2 + (y - y_0)^2} dy &\geq \int_{g(x)}^{f(x)} \sqrt{(x - x_0)^2 + \left(y - \frac{f(x) + g(x)}{2}\right)^2} dy \\ &= \int_{\frac{1}{2}(g(x) - f(x))}^{\frac{1}{2}(f(x) - g(x))} \sqrt{(x - x_0)^2 + y^2} dy. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} \mu_Q^* &= \frac{\int_a^b \int_{g(x)}^{f(x)} \sqrt{(x - x_0)^2 + (y - y_0)^2} dy dx}{\text{area}(Q)} \\ &\geq \frac{\int_a^b \int_{\frac{1}{2}(g(x) - f(x))}^{\frac{1}{2}(f(x) - g(x))} \sqrt{(x - x_0)^2 + y^2} dy dx}{\text{area}(S(Q, x))} = \mu_{S(Q, \ell_x)}((x_0, 0)) \geq \mu_{S(Q, \ell_x)}^*. \end{aligned}$$

$\square$

**Triangles.** We next consider right triangles of a special kind, lying in the first quadrant, and show that the average distance from the origin to their points is larger than  $\frac{1}{3}$ .

**Lemma 3.** *Let  $T$  a right triangle in the first quadrant based on the  $x$ -axis, with vertices  $(a, 0)$ ,  $(a, b)$ , and  $(1, 0)$ , where  $0 \leq a < 1$ , and  $b > 0$ . Then  $\mu_T(o) > \frac{1}{3}$ .*

**Proof.** We use the simple fact that the  $x$ -coordinate of a point is a lower bound to the distance from the origin.

$$\begin{aligned} \mu_T(o) &= \frac{\int_a^1 (\int_0^{b(1-x)/(1-a)} \sqrt{x^2 + y^2} \, dy) \, dx}{b(1-a)/2} > \frac{\int_a^1 (\int_0^{b(1-x)/(1-a)} x \, dy) \, dx}{b(1-a)/2} \\ &= \frac{\frac{b}{1-a} \int_a^1 x(1-x) \, dx}{b(1-a)/2} = \frac{2}{(1-a)^2} \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_a^1 \\ &= \frac{2}{(1-a)^2} \cdot \frac{(2a^3 - 3a^2 + 1)}{6} = \frac{2}{(1-a)^2} \cdot \frac{(1-a)(1+a-2a^2)}{6} \\ &= \frac{1}{(1-a)} \cdot \frac{(1+a-2a^2)}{3} \geq \frac{1}{3}. \end{aligned}$$

The last inequality in the chain follows from  $0 \leq a < 1$ . The inequality in the lemma is strict, since  $\sqrt{x^2 + y^2} > x$  for all points above the  $x$ -axis.  $\square$

**Corollary 1.** *Let  $P$  be any rhombus. Then  $\mu_P^* > \frac{1}{6} \cdot \Delta(P)$ .*

**Proof.** Without loss of generality, we may assume that  $P$  is symmetric with respect to both the  $x$ -axis and the  $y$ -axis. Let us denote the vertices of  $P$  by  $(-1, 0)$ ,  $(1, 0)$ ,  $(0, -b)$ , and  $(0, b)$ , where  $b \leq 1$ . We have  $\Delta(P) = 2$ . By symmetry,  $\mu_P^*$  equals the average distance between the origin  $(0, 0)$  and the points in one of the four congruent right triangles forming  $P$ . Consider the triangle  $T$  in the first quadrant. By Lemma 3 (with  $a = 0$ ), we have  $\mu_P^* = \mu_T(o) > \frac{1}{3}$ . Since  $\Delta(P) = 2$ , we have  $\mu_P^* > \frac{1}{6} \cdot \Delta(P)$ , as desired.  $\square$

**Lemma 4.** *Let  $T$  be a triangle in the first quadrant with a vertical side on the line  $x = a$ , where  $0 \leq a < 1$ , and a third vertex at  $(1, 0)$ . Then  $\mu_T(o) > \frac{1}{3}$ .*

**Proof.** Refer to Fig. 2(ii). Let  $U$  be a right triangle obtained from  $T$  by translating each vertical chord of  $T$  down until its lower endpoint is on the  $x$ -axis. Note that  $\text{area}(T) = \text{area}(U)$ . Observe also that the average distance from the origin decreases in this transformation, namely  $\mu_T(o) \geq \mu_U(o)$ . By Lemma 3, we have  $\mu_U(o) > \frac{1}{3}$ , and so  $\mu_T(o) > \frac{1}{3}$ , as desired.  $\square$

We now have all necessary ingredients to prove Theorem 1.

**Proof of Theorem 1.** Refer to Fig. 2. Let  $Q$  be a convex body in the plane, and let  $c, d \in Q$  be two points at  $\Delta(Q)$  distance apart. We may assume that  $c = (-1, 0)$  and  $d = (1, 0)$ , by a similarity transformation if necessary, so that  $\Delta(Q) = 2$  (the ratio  $\mu_Q^*/\Delta(Q)$  is invariant under similarities). Apply a Steiner symmetrization with respect to the  $x$ -axis, and then a second Steiner symmetrization with respect to the  $y$ -axis. The resulting body  $Q' = S(S(Q, \ell_x), \ell_y)$  is convex, and it is symmetric with respect to both coordinate axes. We have  $\Delta(Q') = \Delta(Q) = 2$  by Lemma 1, and in fact  $c, d \in Q'$ . We also have  $\mu_{Q'}^* \leq \mu_Q^*$  by Lemma 2.

Let  $Q_1$  be the part of  $Q'$  lying in the first quadrant:  $Q_1 = \{(x, y) \in Q' : x, y \geq 0\}$ . By symmetry,  $FW_{Q'} = o$  and we have  $\mu_{Q'}^* = \mu_{Q'}(o) = \mu_{Q_1}(o)$ . Let  $\gamma$  be the portion of the boundary of  $Q'$  lying

in the first quadrant, between points  $b = (0, h)$ , with  $0 < h \leq 1$ , and  $d = (1, 0)$ . For any two points  $p, q \in \gamma$  along  $\gamma$ , denote by  $\gamma(p, q)$  the portion of  $\gamma$  between  $p$  and  $q$ . Let  $r$  be the intersection point of  $\gamma$  and the vertical line  $x = \frac{1}{3}$ .

For a positive integer  $n$ , subdivide  $Q_1$  into at most  $2n + 2$  pieces as follows. Choose  $n + 1$  points  $b = q_1, q_2, \dots, q_{n+1} = r$  along  $\gamma(b, r)$  such that  $q_i$  is the intersection of  $\gamma$  and the vertical line  $x = (i - 1)/3n$ . Connect each of the  $n + 1$  points to  $d$  by a straight line segment. These segments subdivide  $Q_1$  into  $n + 2$  pieces: the right triangle  $T_0 = \Delta bod$ ; a convex body  $Q_0$  bounded by  $rd$  and  $\gamma(r, d)$ ; and  $n$  curvilinear triangles  $\Delta q_i dq_{i+1}$  for  $i = 1, 2, \dots, n$ . For simplicity, we assume that neither  $Q_0$ , nor any of the curvilinear triangles are degenerate; otherwise they can be safely ignored (they do not contribute to the value of  $\mu_Q^*$ ). Subdivide each curvilinear triangle  $\Delta q_i dq_{i+1}$  along the vertical line through  $q_{i+1}$  into a small curvilinear triangle  $S_i$  on the left and a triangle  $T_i$  incident to point  $d$  on the right. The resulting subdivision has  $2n + 2$  pieces, under the nondegeneracy assumption.

By Lemma 3, we have  $\mu_{T_0}(o) > \frac{1}{3}$ . Observe that the difference  $\mu_{T_0}(o) - \frac{1}{3}$  does not depend on  $n$ , and let  $\delta = \mu_{T_0}(o) - \frac{1}{3}$ . By Lemma 4, we also have  $\mu_{T_i}(o) > \frac{1}{3}$ , for each  $i = 1, 2, \dots, n$ . Since every point in  $Q_0$  is at distance at least  $\frac{1}{3}$  from the origin, we also have  $\mu_{Q_0}(o) \geq \frac{1}{3}$ .

For the  $n$  curvilinear triangles  $S_i$ ,  $i = 1, 2, \dots, n$ , we use the trivial lower bound  $\mu_{S_i}(o) \geq 0$ . We now show that their total area  $s_n = \sum_{i=1}^n \text{area}(S_i)$  tends to 0 if  $n$  goes to infinity. Recall that the  $y$ -coordinates of the points  $q_i$  are at most 1, and their  $x$ -coordinates are at most  $\frac{1}{3}$ . This implies that the slope of every line  $q_i d$ ,  $i = 1, 2, \dots, n + 1$ , is in the interval  $[-3/2, 0]$ . Therefore,  $S_i$  is contained in a right triangle bounded by a horizontal line through  $q_i$ , a vertical line through  $q_{i+1}$ , and the line  $q_i d$ . The area of this triangle is at most  $\frac{1}{2}(\frac{1}{3n} \cdot (\frac{3}{2} \cdot \frac{1}{3n})) = 1/(12n^2)$ . That is,  $s_n = \sum_{i=1}^n \text{area}(S_i) \leq 1/(12n)$ . In particular,  $s_n \leq \delta \cdot \text{area}(T_0)$  for a sufficiently large  $n$ . Then we can write

$$\begin{aligned} \mu_{Q_1}(o) &= \frac{\int_{p \in Q_1} \text{dist}(o, p) \, dp}{\text{area}(Q_1)} \geq \frac{\mu_{Q_0}(o) \cdot \text{area}(Q_0) + \sum_{i=0}^n \mu_{T_i}(o) \cdot \text{area}(T_i)}{\text{area}(Q_1)} \\ &\geq \frac{\frac{1}{3}(\text{area}(Q_1) - s_n) + \delta \cdot \text{area}(T_0)}{\text{area}(Q_1)} \geq \frac{1}{3} + \frac{2\delta \cdot \text{area}(T_0)}{3 \cdot \text{area}(Q_1)} > \frac{1}{3}. \end{aligned}$$

This concludes the proof of Theorem 1.  $\square$

**Remark.** A finite triangulation, followed by taking the limit suffices to prove the slightly weaker, non-strict inequality:  $\mu_Q^* \geq \frac{1}{6} \cdot \Delta(Q)$ .

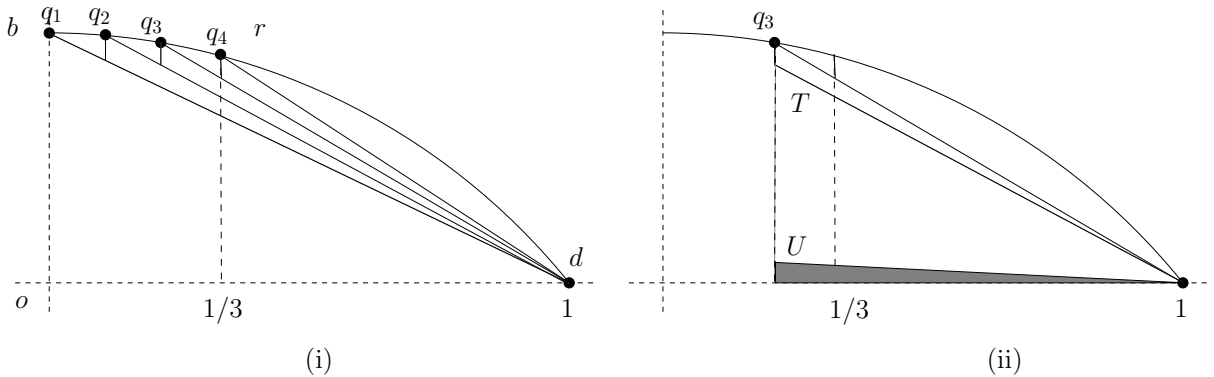


Figure 2: (i) The subdivision of  $Q_1$  for  $n = 3$ . Here  $o = (0, 0)$ ,  $q_1 = b = (0, h)$ ,  $q_4 = r$ ,  $d = (1, 0)$ . (ii) Transformation in the proof of Lemma 4.

### 3 Upper bounds: proof of Theorem 2

Let  $Q$  be a planar convex body and let  $D = \Delta(Q)$ . Let  $\partial Q$  denote the boundary of  $Q$ , and let  $\text{int}(Q)$  denote the interior of  $Q$ . Let  $\Omega$  be the smallest disk enclosing  $Q$ , and let  $o$  and  $R$  be the center and respectively the radius of  $\Omega$ . Write  $a = \frac{2(4-\sqrt{3})}{13}$ . By the convexity of  $Q$ ,  $o \in Q$ , as observed in [1]. Moreover, Abu-Affash and Katz [1] have shown that the average distance from  $o$  to the points in  $Q$  satisfies

$$\mu_Q(o) \leq \frac{2}{3\sqrt{3}} \cdot \Delta(Q) < 0.3850 \cdot \Delta(Q).$$

Here we further refine their analysis and derive a better upper bound on the average distance from  $o$  to the points in  $Q$ :

$$\mu_Q(o) \leq \frac{2(4-\sqrt{3})}{13} \cdot \Delta(Q) < 0.3490 \cdot \Delta(Q).$$

Since the average distance from the Fermat-Weber center of  $Q$  is not larger than that from  $o$ , we immediately get the same upper bound on  $c_2$ . We need the next simple lemma established in [1]. Its proof follows from the definition of average distance.

**Lemma 5.** [1]. *Let  $Q_1$  and  $Q_2$  be two (not necessarily convex) disjoint bodies in the plane, and  $p$  be a point in the plane. Then  $\mu_{(Q_1 \cup Q_2)}(p) \leq \max(\mu_{Q_1}(p), \mu_{Q_2}(p))$ .*

By induction, Lemma 5 yields the following:

**Lemma 6.** *Let  $Q_1, Q_2, \dots, Q_n$  be  $n$  (not necessarily convex) pairwise disjoint bodies in the plane, and  $p$  be a point in the plane. Then*

$$\mu_{(Q_1 \cup \dots \cup Q_n)}(p) \leq \max(\mu_{Q_1}(p), \dots, \mu_{Q_n}(p)).$$

We also need the following classical result of Jung [10]; see also [9].

**Theorem 3.** (Jung [10]). *Let  $S$  be a set of diameter  $\Delta(S)$  in the plane. Then  $S$  is contained in a disk of radius  $\frac{1}{\sqrt{3}} \cdot \Delta(S)$ .*

By Theorem 3 we have

$$\frac{1}{2}D \leq R \leq \frac{1}{\sqrt{3}}D. \tag{1}$$

Consider a circular sector of radius  $r$  and central angle  $\alpha$ . Observe that the average distance from the center of the circle to the points in the sector is

$$\frac{\int_0^r \alpha x^2 dx}{\int_0^r \alpha x dx} = \frac{\alpha r^3/3}{\alpha r^2/2} = \frac{2r}{3}. \tag{2}$$

**Proof of Theorem 2.** If  $o \in \partial Q$  then  $Q$  is contained in a halfdisk  $\Theta$  of  $\Omega$ , of the same diameter  $D$ , with  $o$  as the midpoint of this diameter. Then by (2), it follows that  $\mu_Q(o) \leq \frac{1}{3} \cdot D$ , as required.

We can therefore assume that  $o \in \text{int}(Q)$ . Let  $\varepsilon > 0$  be sufficiently small. For a large positive integer  $n$ , subdivide  $\Omega$  into  $n$  congruent circular double sectors (wedges)  $W_1, \dots, W_n$ , symmetric about  $o$  (the center of  $\Omega$ ), where each sector subtends an angle  $\alpha = \pi/n$ . Consider a double sector  $W_i = U_i \cup V_i$ , where  $U_i$  and  $V_i$  are circular sectors of  $\Omega$ . Let  $X_i \subseteq U_i$ , and  $Y_i \subseteq V_i$  be two minimal circular sectors centered at  $o$  and containing  $U_i \cap Q$ , and  $V_i \cap Q$ , respectively:  $U_i \cap Q \subseteq X_i$ , and

$V_i \cap Q \subseteq Y_i$ . Let  $x_i$  and  $y_i$  be the radii of  $X_i$  and  $Y_i$ , respectively. Let  $X'_i \subseteq X_i$ , and  $Y'_i \subseteq Y_i$  be two circular subsectors of radii  $(1 - \varepsilon)x_i$  and  $(1 - \varepsilon)y_i$ , respectively. Since  $o \in \text{int}(Q)$ , we can select  $n = n(Q, \varepsilon)$  large enough, so that for each  $1 \leq i \leq n$ , the subsectors  $X'_i$  and  $Y'_i$  are nonempty and entirely contained in  $Q$ . That is, for every  $i$ , we have

$$X'_i \cup Y'_i \subseteq W_i \cap Q \subseteq X_i \cup Y_i.$$

It is enough to show that for any double sector  $W = W_i$ , we have

$$\lim_{\varepsilon \rightarrow 0} \mu_{(W \cap Q)}(o) \leq aD,$$

since then, Lemma 6 (with  $W_i$  being the  $n$  pairwise disjoint regions) will imply that  $\mu_Q(o) \leq aD$ , concluding the proof of Theorem 2. For simplicity, write  $x = x_i$ , and  $y = y_i$ . Obviously the diameter of  $W \cap Q$  is at most  $D$ , hence  $x + y \leq D$ . We can assume w.l.o.g. that  $y \leq x$ , so by Theorem 3 we also have  $x \leq \frac{1}{\sqrt{3}} \cdot D$ . Hence so far, our constraints are:

$$0 < y \leq x \leq \frac{1}{\sqrt{3}} \cdot D \quad \text{and} \quad x + y \leq D. \quad (3)$$

By the minimality of the disk  $\Omega$ , the convex body  $Q$  either contains three points  $q_1, q_2, q_3$  on the boundary of  $\Omega$  such that the triangle  $q_1q_2q_3$  contains the disk center  $o$  in the interior, or contains two points  $q_1, q_2$  on the boundary of  $\Omega$  such that the segment  $q_1q_2$  goes through the disk center  $o$ . In the latter case, the segment  $q_1q_2$  can be viewed as a degenerate triangle  $q_1q_2q_3$  with two coinciding vertices  $q_2$  and  $q_3$ .

Let  $r$  be the radius of the largest disk centered at  $o$  that is contained in the convex body  $Q$ . Then  $r$  is at least the distance from  $o$  to the longest side of the triangle  $q_1q_2q_3$ , say  $q_1q_2$ . Since  $|q_1q_2| \leq D$ ,  $|oq_1| = |oq_2| = R$ , we have

$$r \geq \sqrt{R^2 - D^2/4}.$$

By taking (1) into account, the constraints in (3) can be expanded to the following:

$$\sqrt{R^2 - D^2/4} \leq y \leq x \leq R \leq D/\sqrt{3} \quad \text{and} \quad x + y \leq D. \quad (4)$$

By the definition of average distance, we can write

$$\begin{aligned} \mu_{(W \cap Q)}(o) &= \frac{\int_{p \in (W \cap Q)} \text{dist}(o, p) \, dp}{\text{area}(W \cap Q)} \\ &\leq \frac{\alpha \cdot \frac{x^2}{2} \cdot \frac{2x}{3} + \alpha \cdot \frac{y^2}{2} \cdot \frac{2y}{3}}{\alpha(1 - \varepsilon)^2 \cdot \left(\frac{x^2}{2} + \frac{y^2}{2}\right)} = \frac{2}{3} \cdot \frac{x^3 + y^3}{(1 - \varepsilon)^2 \cdot (x^2 + y^2)}. \end{aligned} \quad (5)$$

Let

$$f(x, y) = \frac{2}{3} \cdot \frac{x^3 + y^3}{x^2 + y^2}, \quad \text{and} \quad f_1(x, y, \varepsilon) = \frac{2}{3} \cdot \frac{x^3 + y^3}{(1 - \varepsilon)^2 \cdot (x^2 + y^2)}. \quad (6)$$

Clearly for any feasible pair  $(x, y)$ , we have

$$\lim_{\varepsilon \rightarrow 0} f_1(x, y, \varepsilon) = f(x, y).$$

Thus

$$\lim_{\varepsilon \rightarrow 0} \mu_{(W \cap Q)}(o) \leq \lim_{\varepsilon \rightarrow 0} f_1(x, y, \varepsilon) = f(x, y). \quad (7)$$

It remains to find an upper bound on  $f(x, y)$  subject to the constraints in (4).



**Lemma 7.** *Subject to the constraints in (4), we have*

$$f(x, y) \leq \frac{2(4 - \sqrt{3})}{13} \cdot D. \quad (8)$$

**Proof.** Throughout our analysis, we may assume that  $D$  is a fixed constant and  $x$ ,  $y$ , and  $R$  are variable parameters. Substituting  $z = y/x$  in (6), we have

$$f(x, y) = g(x, z) = \frac{2x}{3} \cdot \frac{1 + z^3}{1 + z^2}.$$

Then, taking the partial derivative of  $g(x, z)$  with respect to  $z$ , we have

$$\begin{aligned} \frac{\partial}{\partial z} g(x, z) &= \frac{2x}{3} \cdot \left( \frac{3z^2}{1 + z^2} - \frac{1 + z^3}{(1 + z^2)^2} 2z \right) \\ &= \frac{2x}{3} \cdot \frac{3z^2(1 + z^2) - (1 + z^3)2z}{(1 + z^2)^2} = \frac{2x}{3} \cdot \frac{z(z^3 + 3z - 2)}{(1 + z^2)^2}. \end{aligned}$$

The cubic equation  $z^3 + 3z - 2 = 0$  has exactly one real root  $z_0 = (\sqrt{2} + 1)^{1/3} - (\sqrt{2} - 1)^{1/3} = 0.596\dots$ . Thus for a fixed  $x$ , the function  $g(x, z)$  is strictly decreasing for  $0 \leq z \leq z_0$  and is strictly increasing for  $z_0 \leq z \leq 1$ . Therefore, by the upper bound that  $x + y \leq D$  and the lower bound that  $\sqrt{R^2 - D^2/4} \leq r \leq y$  in (4), the function  $f(x, y)$  is maximized when  $y$  takes one of the following two extreme values:

$$y_1 = \sqrt{R^2 - D^2/4} \quad \text{and} \quad y_2 = D - x.$$

By the inequality that  $x \leq R \leq D/\sqrt{3}$  in (4), it follows that  $x + y_1 \leq R + \sqrt{R^2 - D^2/4} \leq D/\sqrt{3} + D/\sqrt{12} < D$ . Since  $x + y_2 = D$ , we have  $y_1 < y_2$ .

**Case 1.** We first consider the easy case that  $y = y_2$ . Then  $x + y = D$ , and we have

$$f(x, y) = \frac{2}{3} \cdot \frac{x^3 + y^3}{x^2 + y^2} = \frac{2}{3} \cdot \frac{(x + y)^3 - 3(x + y)xy}{(x + y)^2 - 2xy} = \frac{2}{3} \cdot \frac{D^3 - 3Dxy}{D^2 - 2xy}.$$

Substituting  $w = xy$ , we transform the function  $f(x, y)$  to a function  $h_1(w)$ :

$$f(x, y) = h_1(w) = \frac{2}{3} \cdot \frac{3Dw - D^3}{2w - D^2}.$$

The function  $h_1(w)$  is decreasing in  $w$  because

$$\begin{aligned} \frac{d}{dw} h_1(w) &= \frac{2}{3} \cdot \left( \frac{3D}{2w - D^2} - \frac{2(3Dw - D^3)}{(2w - D^2)^2} \right) \\ &= \frac{2}{3} \cdot \frac{3D(2w - D^2) - 2(3Dw - D^3)}{(2w - D^2)^2} = \frac{2}{3} \cdot \frac{-D^3}{(2w - D^2)^2} \leq 0. \end{aligned}$$

Thus  $f(x, y)$  is maximized when  $xy$  is minimized. With the sum  $x + y$  fixed at  $D$ , and under the constraint that  $x \leq R \leq D/\sqrt{3}$  in (4), the product  $xy$  is minimized when  $x = \frac{1}{\sqrt{3}}D$  and  $y = \left(1 - \frac{1}{\sqrt{3}}\right)D$ . Thus we have

$$f(x, y) \leq \frac{2}{3} \cdot \frac{\left(\frac{1}{\sqrt{3}}\right)^3 + \left(1 - \frac{1}{\sqrt{3}}\right)^3}{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(1 - \frac{1}{\sqrt{3}}\right)^2} D = \frac{2(4 - \sqrt{3})}{13} D = 0.3489\dots D. \quad (9)$$

**Case 2.** We next consider the case<sup>1</sup> that  $y = y_1$ . With  $y$  fixed, the function  $f(x, y)$  is maximized when  $x$  is as large as possible because

$$\begin{aligned}\frac{\partial}{\partial x}f(x, y) &= \frac{2}{3} \cdot \left( \frac{3x^2}{x^2 + y^2} - \frac{x^3 + y^3}{(x^2 + y^2)^2} 2x \right) \\ &= \frac{2}{3} \cdot \frac{3x^2(x^2 + y^2) - (x^3 + y^3)2x}{(x^2 + y^2)^2} \\ &= \frac{2}{3} \cdot \frac{x(x^3 + 3xy^2 - 2y^3)}{(x^2 + y^2)^2} \\ &\geq \frac{2}{3} \cdot \frac{x(y^3 + 3y^3 - 2y^3)}{(x^2 + y^2)^2} \geq 0.\end{aligned}$$

Thus for  $y = \sqrt{R^2 - D^2/4}$  and under the constraint that  $x \leq R$  in (4), the function  $f(x, y)$  is maximized when  $x = R$  and  $y = \sqrt{R^2 - D^2/4} = \sqrt{x^2 - D^2/4}$ . It follows that

$$\frac{dx}{dR} = 1 \quad \text{and} \quad \frac{dy}{dR} = \frac{d\sqrt{x^2 - D^2/4}}{dR} = \frac{x}{\sqrt{x^2 - D^2/4}} = x/y.$$

Let  $h_2(R) = f(R, \sqrt{R^2 - D^2/4})$ . We next show that  $h_2(R)$  is increasing in  $R$ . Taking the derivative, we have

$$\begin{aligned}\frac{d}{dR}h_2(R) &= \frac{2}{3} \cdot \left( \frac{3x^2 \frac{dx}{dR} + 3y^2 \frac{dy}{dR}}{x^2 + y^2} - \frac{x^3 + y^3}{(x^2 + y^2)^2} \left( 2x \frac{dx}{dR} + 2y \frac{dy}{dR} \right) \right) \\ &= \frac{2}{3} \cdot \left( \frac{3x^2 + 3y^2(x/y)}{x^2 + y^2} - \frac{x^3 + y^3}{(x^2 + y^2)^2} (2x + 2y(x/y)) \right) \\ &= \frac{2}{3} \cdot \frac{(3x^2 + 3xy)(x^2 + y^2) - (x^3 + y^3)(2x + 2x)}{(x^2 + y^2)^2} \\ &= \frac{2}{3} \cdot \frac{(3x^4 + 3x^2y^2 + 3x^3y + 3xy^3) - (4x^4 + 4xy^3)}{(x^2 + y^2)^2} \\ &= \frac{2}{3} \cdot \frac{(x^4 + 3x^2y^2 + 3x^3y + xy^3) - (2x^4 + 2xy^3)}{(x^2 + y^2)^2} \\ &= \frac{2}{3} \cdot \frac{x^4}{(x^2 + y^2)^2} \cdot ((1 + y/x)^3 - 2 - 2(y/x)^3).\end{aligned}$$

Substituting  $z = y/x$ , we simplify the last factor  $(1 + y/x)^3 - 2 - 2(y/x)^3$  in the resulting expression above to

$$h_3(z) = (1 + z)^3 - 2 - 2z^3.$$

To show that  $\frac{d}{dR}h_2(R) > 0$ , it remains to show that  $h_3(z) > 0$ . For  $0 \leq z \leq 1$ , the function  $h_3(z)$  is increasing in  $z$  because

$$\frac{d}{dz}h_3(z) = 3(1 + z)^2 - 6z^2 = -3(1 - z)^2 + 6 \geq 6 - 3 > 0.$$

Recall that  $x \geq y$ . If  $R \leq \frac{3(4-\sqrt{3})}{13}D$ , then we would easily have

$$f(x, y) = \frac{2}{3} \cdot \frac{x^3 + y^3}{x^2 + y^2} \leq \frac{2}{3} \cdot \frac{x^3}{x^2} = \frac{2}{3}x \leq \frac{2}{3}R \leq \frac{2(4 - \sqrt{3})}{13}D,$$

---

<sup>1</sup>This case, when  $x + y < D$ , has been mistakenly overlooked in the proof given in [6].

which matches the upper bound in case 1. Now suppose that  $R > \frac{3(4-\sqrt{3})}{13}D$ . Then

$$D/R < \frac{13}{3(4-\sqrt{3})} \text{ and } z = y/x = \sqrt{1 - (D/R)^2/4} > \sqrt{1 - \left(\frac{13}{3(4-\sqrt{3})}\right)^2/4} = 0.2955\dots$$

It follows that

$$h_3(z) > h_3\left(\sqrt{1 - \left(\frac{13}{3(4-\sqrt{3})}\right)^2/4}\right) = 0.1226\dots > 0,$$

hence

$$\frac{d}{dR}h_2(R) > 0.$$

We have shown that the function  $h_2(R)$  is increasing in  $R$ . Then, under the constraint that  $R \leq D/\sqrt{3}$  in (4),  $h_2(R)$  is maximized when  $R = \frac{1}{\sqrt{3}}D$ . Correspondingly,  $f(x, y)$  is maximized when  $x = \frac{1}{\sqrt{3}}D$  and  $y = \frac{1}{\sqrt{12}}D$ . Thus

$$f(x, y) \leq \frac{2}{3} \cdot \frac{\left(\frac{1}{\sqrt{3}}\right)^3 + \left(\frac{1}{\sqrt{12}}\right)^3}{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{12}}\right)^2} D = \frac{\sqrt{3}}{5} D = 0.3464\dots D, \quad (10)$$

which is (slightly) smaller than the upper bound obtained in case 1. This proves the lemma.  $\square$

By (7) and (8) we deduce that

$$\lim_{\varepsilon \rightarrow 0} \mu_{(W \cap Q)}(o) \leq \lim_{\varepsilon \rightarrow 0} f_1(x, y, \varepsilon) = f(x, y) \leq \frac{2(4-\sqrt{3})}{13} \cdot D,$$

as required. This completes the proof for the case of arbitrary convex bodies.

**Centrally symmetric body.** Assume now that  $Q$  is centrally symmetric with respect to a point  $q$ . We repeat the same ‘‘double sector’’ argument. It is enough to observe that: (i) the center of  $\Omega$  coincides with  $q$ , that is,  $o = q$ ; and (ii)  $x = y \leq \frac{1}{2} \cdot D$  for any double sector  $W$ . By (5), the average distance calculation yields now

$$\mu_{(W \cap Q)}(o) \leq \frac{2x^3}{3(1-\varepsilon)^2 \cdot x^2} = \frac{2x}{3(1-\varepsilon)^2} \leq \frac{D}{3(1-\varepsilon)^2},$$

and by taking the limit when  $\varepsilon$  tends to zero, we obtain

$$\mu_Q(o) \leq \frac{D}{3},$$

as required. The proof of Theorem 2 is now complete.  $\square$

## 4 Applications

1. Carmi, Har-Peled and Katz [3] showed that given a convex polygon  $Q$  with  $n$  vertices, and a parameter  $\varepsilon > 0$ , one can compute an  $\varepsilon$ -approximate Fermat-Weber center  $q \in Q$  in  $O(n + 1/\varepsilon^4)$  time such that  $\mu_Q(q) \leq (1 + \varepsilon)\mu_Q^*$ . Abu-Affash and Katz [1] gave a simple  $O(n)$ -time algorithm for computing the center  $q$  of the smallest disk enclosing  $Q$ , and showed that  $q$  approximates the

Fermat-Weber center of  $Q$ , with  $\mu_Q(q) \leq \frac{25}{6\sqrt{3}}\mu_Q^*$ . Our Theorems 1 and 2, combined with their analysis, improves the approximation ratio to about 2.09:

$$\mu_Q(q) \leq \frac{12(4 - \sqrt{3})}{13}\mu_Q^*.$$

**2.** The value of the constant  $c_1$  (i.e., the infimum of  $\mu_Q^*/\Delta(Q)$  over all convex bodies  $Q$  in the plane) plays a key role in the following load balancing problem introduced by Aronov, Carmi and Katz [2]. We are given a convex body  $D$  and  $m$  points  $p_1, p_2, \dots, p_m$  representing *facilities* in the interior of  $D$ . Subdivide  $D$  into  $m$  convex regions,  $R_1, R_2, \dots, R_m$ , of equal area such that  $\sum_{i=1}^m \mu_{p_i}(R_i)$  is minimal. Here  $\mu_{p_i}(R_i)$  is the *cost* associated with facility  $p_i$ , which may be interpreted as the average travel time from the facility to any location in its designated region, each of which has the same area. One of the main results in [2] is a  $(8 + \sqrt{2\pi})$ -factor approximation in the case that  $D$  is an  $n_1 \times n_2$  rectangle for some integers  $n_1, n_2 \in \mathbb{N}$ . This basic approximation bound is then used for several other cases, e.g., subdividing a convex fat domain  $D$  into  $m$  convex regions  $R_i$ .

By substituting  $c_1 = \frac{1}{6}$  (Theorem 1) into the analysis in [2], the upper bound for the approximation ratio improves from  $8 + \sqrt{2\pi} \approx 10.5067$  to  $7 + \sqrt{2\pi} \approx 9.5067$ . It can be further improved by optimizing another parameter used in their calculation. Let  $S$  be a unit square and let  $s \in S$  be an arbitrary point in the square. Aronov et al. [2] used the upper bound  $\mu_S(s) \leq \frac{2}{3}\sqrt{2} \approx 0.9429$ . It is clear that  $\max_{s \in S} \mu_S(s)$  is attained if  $s$  is a vertex of  $S$ . The average distance of  $S$  from such a vertex, say  $v$ , is  $\mu_S(v) = \frac{1}{3}(\sqrt{2} + \ln(1 + \sqrt{2})) \approx 0.7652$ , and so  $\mu_S(s) \leq \frac{1}{3}(\sqrt{2} + \ln(1 + \sqrt{2}))$ , for any  $s \in S$ . With these improvements, the upper bound on the approximation ratio becomes  $7 + \frac{\sqrt{\pi}}{2}(\sqrt{2} + \ln(1 + \sqrt{2})) \approx 9.0344$ .

## 5 Conclusion

We have shown that  $\mu_Q^* \leq \frac{2(4-\sqrt{3})}{13} \cdot \Delta(Q)$  for any convex body  $Q$  in the plane, and thus stopped short of proving the inequality conjectured by Carmi, Har-Peled and Katz [3]:

**Conjecture 1.** [3]. *For any planar convex body  $Q$  of diameter  $D$ , the average distance from the Fermat-Weber center of  $Q$  to the points in  $Q$  is at most  $D/3$ :*

$$\mu_Q^* \leq \frac{1}{3} \cdot \Delta(Q). \tag{11}$$

A small further improvement to our bound can be obtained as follows. Recall the argument in Section 3. Let  $r(t)$  be the distance from  $o$  to the boundary of  $Q$  in direction  $t \in [0, 2\pi)$ . With the  $x, y$ , notation we used there, we have  $x(t) = \max(r(t), r(t + \pi))$  and  $y(t) = \min(r(t), r(t + \pi))$ .

We are interested in bounding

$$\frac{\int_0^{2\pi} r^2 dt}{\int_0^\pi r dt} = \frac{\int_0^\pi (x^2 + y^2) dt}{\int_0^\pi (x + y) dt} = \frac{\int_0^\pi (r(t)^2 + r(t + \pi)^2) dt}{\int_0^\pi (r(t) + r(t + \pi)) dt}.$$

If  $R$  denotes the radius of the disk  $\Omega$ , then we have  $r(t) = R$  for at least three distinct values  $t \in [0, 2\pi)$ , corresponding to the three vertices  $q_1, q_2, q_3$  on the boundary of  $\Omega$ . Since  $r(t)$  is continuous, we have  $r(t) = r(t + \pi)$  for at least three distinct values  $t_1, t_2, t_3 \in [0, 2\pi)$ , by the mean value theorem. We have shown that  $Q$  contains a small disk of radius  $\sqrt{R^2 - D^2/4}$ . If this radius is not too small, then  $r(t)$  cannot change too rapidly, and there will be at least three

short intervals in the circular interval  $[0, 2\pi)$ , where  $|x(t) - y(t)| \leq \varepsilon$  for some small  $\varepsilon$ . In any such interval,  $y(t) \leq x(t) \leq D/2 + \varepsilon/2$ , and this leads to a small improvement by averaging. On the other hand, if the radius  $\sqrt{R^2 - D^2/4}$  is too small, i.e.,  $R$  is close to  $D/2$ , then we can use this fact for a slightly better bound, as in the current proof. Our preliminary calculations show however that the sought improvement would only affect the 4th digit in the upper bound, and so we decided to omit the details. Obtaining more substantial improvements or closing the remaining gap probably requires new ideas.

We conclude this discussion by outlining two possible avenues (Conjectures 2 and 3 below) for proving Conjecture 1.

**Conjecture 2.** *For any planar convex body  $Q$  of diameter  $D$ , the average distance from the center  $o$  of the circumscribed disk  $\Omega$  of  $Q$  to the points in  $Q$  is at most  $D/3$ :*

$$\mu_Q(o) \leq \frac{1}{3} \cdot \Delta(Q). \quad (12)$$

Let  $Q$  be a planar convex body. Let  $Q' = \frac{1}{2}(Q - Q)$  be the convex body obtained from  $Q$  by Steiner symmetrization with respect to a point; see [12, Exercise 6-9]. It is well known that (i)  $Q'$  is centrally symmetric, (ii)  $\text{area}(Q) \leq \text{area}(Q')$ , (iii)  $\Delta(Q) = \Delta(Q')$ . Also  $Q' = Q$  if  $Q$  is centrally symmetric. Another useful property of this transformation might be that it does not decrease the average distance from the Fermat-Weber center:

**Conjecture 3.** *The average distance from the Fermat-Weber center of  $Q$  to the points in  $Q$  is at most the average distance from the Fermat-Weber center of  $Q'$  to the points in  $Q'$ :*

$$\mu_Q(FW_Q) \leq \mu_{Q'}(FW_{Q'}). \quad (13)$$

Our Theorem 2 confirms Conjecture 1 for centrally symmetric convex bodies. Since the diameter of a convex body is preserved by Steiner symmetrization, verifying Conjecture 3 would be enough to prove Conjecture 1.

We conclude with an example. Recall the three classical centers in the geometry of planar convex bodies, besides the Fermat-Weber center:

1. the center of the smallest circumscribed disk: point  $o$  in Fig. 3.
2. the center of the largest inscribed disk: point  $i$  in Fig. 3.
3. the center of mass (or gravity): point  $g$  in Fig. 3.
4. the Fermat-Weber center: point  $f = FW_Q$  in Fig. 3.

In general these four centers are distinct. Refer to Fig. 3. Let  $Q$  be a right isosceles triangle with its vertices at  $(-1, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ . Then  $Q' = \frac{1}{2}(Q - Q)$  is a centrally symmetric hexagon centered at  $(0, 0)$ . Both  $Q$  and  $Q'$  have the same diameter  $D = 2$ . The four centers of  $Q$  are located at  $(0, y)$  for different values of  $y$ , corresponding to different average distances:

$$\begin{array}{ll} y_i = \sqrt{2} - 1 \approx 0.4142 & \mu_Q(i) \approx 0.2149 \cdot D \\ y_f \approx 0.3511 & \mu_Q(f) \approx 0.2130 \cdot D \\ y_g = 1/3 \approx 0.3333 & \mu_Q(g) \approx 0.2132 \cdot D \\ y_o = 0 & \mu_Q(o) \approx 0.2705 \cdot D \end{array}$$

Observe that  $\mu_Q(o) < D/3$ , which is consistent with Conjecture 2.

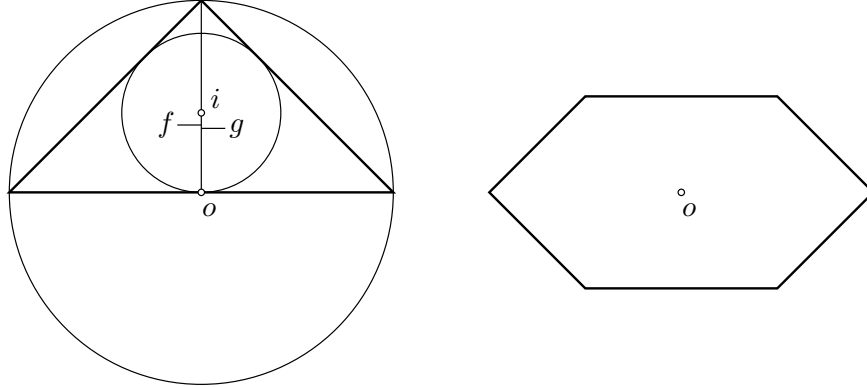


Figure 3: Left: the four centers of the triangle  $Q$  are distinct. Right: the four centers of the hexagon  $Q' = \frac{1}{2}(Q - Q)$  coincide.

The four centers of  $Q'$  coincide at the same point  $o$ . A simple geometric argument shows that  $\mu_{Q'}(o)/\mu_Q(o) = (\sqrt{1/2} + 2)/3$ . Therefore we have

$$\mu_{Q'}(o) \approx 0.2441 \cdot D.$$

Observe that  $\mu_Q(f) < \mu_{Q'}(o)$ , which is consistent with Conjecture 3.

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