

On convexification of polygons by pops

Adrian Dumitrescu* Evan Hilscher †

November 20, 2009

Abstract

Given a polygon P in the plane, a *pop* operation is the reflection of a vertex with respect to the line through its adjacent vertices. We define a family of alternating polygons, and show that any polygon from this family cannot be convexified by pop operations. This family contains simple, as well as non-simple (i.e., self-intersecting) polygons, as desired. We thereby answer in the negative an open problem posed by Demaine and O’Rourke [9, Open Problem 5.3].

Keywords: Polygon convexification, edge-length preserving transformation, pop operation.

1 Introduction

Consider a polygon $P = \{p_1, \dots, p_n\}$ in the plane, that could be simple or self-intersecting. A *pop* operation is the reflection of a vertex, say p_i , with respect to the line through its adjacent vertices p_{i-1} and p_{i+1} (as usual indexes are taken modulo n , i.e., $p_{n+1} = p_1$) [5]. Observe that for the operation to be well-defined we need that p_{i-1} and p_{i+1} are distinct. This operation belongs to the larger class of edge-length preserving transformations, when applied to polygons [5, 19, 20, 21, 22]. It seems to have been used for the first time by Millet [16]. If instead of reflecting p_i with respect to the line through its adjacent vertices p_{i-1} and p_{i+1} , the reflection is executed with respect to the midpoint of p_{i-1} and p_{i+1} , the operation is called a *popturn*; see [4, 5]. Observe that both the pop and the popturn are single-vertex operations.

Each is an instance of a “flip”, defined informally, which has been studied at length. The most common variant of flip is the *pocket flip* (or just *flip*), first considered by Erdős [10]. Another variant is the *flipturn*, first considered by Kazarinoff, and later by Joss and Shannon; see [8, 12] for an account of their results. In contrast with pops and popturns, both the flip and the flipturn may involve multiple vertices. The inverse of a pocket flip, called *deflation*, has been also considered [7, 11]. We briefly describe pocket flips and pocket flipturns next.

Assume that we deal with simple polygons in this paragraph. A *pocket* is a region exterior to the polygon but interior to its convex hull, bounded by a subchain of the polygon edges and the pocket *lid*, the edge of the convex hull connecting the endpoints of that subchain; see e.g., [9, p. 74]. Observe that any non-convex polygon has at least one pocket. A *flip* of a pocket consists of reflecting the pocket about the line through the pocket lid. Instead, a *flipturn* of a pocket consists of reflecting the pocket about the midpoint of the pocket lid. Observe that if P is simple and non-convex, the

*Department of Computer Science, University of Wisconsin–Milwaukee, WI 53201-0784, USA. Email: ad@cs.uwm.edu. Supported in part by NSF CAREER grant CCF-0444188.

†Department of Computer Science, University of Wisconsin–Milwaukee, WI 53201-0784, USA. Email: hilscher@uwm.edu.

polygons resulting after a pocket flip, or a pocket floipturn are again simple. It is known that within both of these variants, convexification can be achieved. More precisely: given a simple polygon, it can be convexified by a finite sequence of pocket flips [8, 12, 13, 15, 17, 18, 23, 24, 25]. Similarly, it can be convexified by a finite sequence of pocket floipturns [12]. Moreover, the first result continues to hold for self-intersecting polygons, under broad assumptions, see [8]. While the convexifying sequence can be arbitrarily long for pocket flips (i.e., irrespective of n , the number of vertices), a quadratic number of operations always suffices in the case of floipturns [1, 3, 6]. There is an extensive bibliography pertaining to these subjects [1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 17, 18, 23, 24, 25]. See also [5, 19, 20, 21, 22] for more results on edge-length preserving transformations and chord stretching.

In this paper we focus on pop operations. Thurston gave an example of a simple polygon that becomes self-intersecting with any pop, see [9, p.81]. Ballinger and Thurston showed (according to [9, p. 81]) that almost any simple polygon can be convexified by pops if self-intersection is permitted; however no proof has been published. As Ballinger writes in his thesis [5], “pops are very natural transformations to consider, but the analysis of polygon convexification by pops seems very tricky”. It has remained an open problem whether there exist polygons that cannot be convexified by pops [9, Open Problem 5.3]. We show here that such polygons do indeed exist, from both classes, simple or self-intersecting, thereby answering the above open problem in its full generality.

In Section 2, for every even $n \geq 6$, we define a family \mathcal{A}_n of *alternating polygons*, and show that any polygon from this family cannot be convexified by pop operations. This family contains simple, as well as non-simple (i.e., self-intersecting) polygons, as desired. It is interesting that this family is closed under pop operations: any pop operation applied to a polygon in \mathcal{A}_n , at any vertex, yields a polygon in \mathcal{A}_n .

2 Alternating polygons

Recall that in order for the pop operation on a vertex p_i be well defined, its neighbors, p_{i-1} and p_{i+1} need to be distinct, so that the reflection line through them is unique, hence the reflection of p_i is also unique. A condition on the edge lengths of the polygon that guarantees this is that no two edges have the same length; such a polygon is called *scalene* [5, p. 24]. A weaker condition that suffices is that no two consecutive edges have the same length; we call such polygons *weakly scalene*. Our family of polygons \mathcal{A}_n we define below consists of weakly scalene polygons.

If p_{i-1} and p_{i+1} coincide, p_i is called a *hairpin vertex* [4]. Popping a hairpin vertex is undefined because there are an infinite number of reflection lines through p_{i-1} and p_{i+1} . Our family of polygons is specifically designed to avoid any occurrence of hairpin vertices. See [4] for a possible adaptation of pops to hairpin vertices.

Let n be even. Fix a coordinate system in the plane. We say that a polygon $P = \{p_1, p_2, \dots, p_n\}$ with n distinct vertices is *alternating* if its vertices lie alternately on the two axes: say, the vertices with *odd* indexes on the x -axis, and the vertices with *even* indexes on the y -axis. See Fig. 1 for an illustration.

Let $n = 2k$. Let $\mathbf{x} = (x_1, x_2, \dots, x_k)$, and $\mathbf{y} = (y_1, y_2, \dots, y_k)$ be two vectors in the positive orthant of \mathbb{R}^k , each having distinct nonzero coordinates, that is:

$$\begin{aligned} i \in \{1, 2, \dots, k\} &\Rightarrow x_i > 0 \text{ and } y_i > 0, \\ i, j \in \{1, 2, \dots, k\} \text{ and } i \neq j &\Rightarrow x_i \neq x_j \text{ and } y_i \neq y_j. \end{aligned}$$

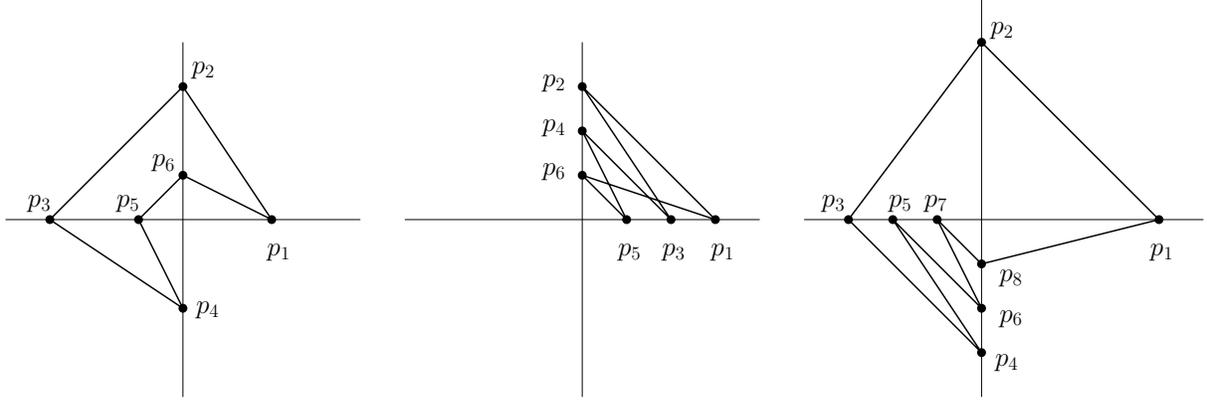


Figure 1: Alternating polygons with 6 and 8 vertices: $A((2, 3, 1), (3, 2, 1), (+1, +1, -1, -1, -1, +1))$, $A((3, 2, 1), (3, 2, 1), (+1, +1, +1, +1, +1, +1))$, and $A((4, 3, 2, 1), (4, 3, 2, 1), (+1, +1, -1, -1, -1, -1, -1, -1))$. The one in the middle is self-intersecting.

Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{2k}) \in \{-1, +1\}^{2k}$ be a binary sign vector. Consider the alternating polygon $A(\mathbf{x}, \mathbf{y}, \sigma) = \{p_1, p_2, \dots, p_{2k}\}$, where

- $p_{2i+1} = (\sigma_{2i+1} \cdot x_{i+1}, 0)$, for $i = 0, \dots, k - 1$.
- $p_{2i} = (0, \sigma_{2i} \cdot y_i)$, for $i = 1, \dots, k$.

Let $\mathcal{A}_n (\equiv \mathcal{A}_{2k})$ be the family of all alternating polygons $A(\mathbf{x}, \mathbf{y}, \sigma)$ defined as above. First note that \mathcal{A}_n contains both simple, as well as non-simple (i.e., self-intersecting) polygons.

Indeed, consider the polygon P_1 described next. Let $x_1 = y_1 = k$, and $x_i = y_i = k - i + 1$, for $i = 2, \dots, k$. Let $\sigma = (+1, +1, -1, \dots, -1)$. It is easy to see that $P_1 \in \mathcal{A}_n$ is a simple polygon. An example is shown in Fig. 1 (right).

Consider now the polygon P_2 described as follows. Let $x_i = y_i = k - i + 1$, for $i = 1, \dots, k$. Let $\sigma = (+1, \dots, +1)$. It is easy to see that $P_2 \in \mathcal{A}_n$ is a self-intersecting polygon. An example is shown in Fig. 1 (middle).

A sequence of pops executed on an alternating simple polygon with 6 vertices appears in Fig. 2. A key fact regarding alternating polygons is the following:

Lemma 1. *If $P \in \mathcal{A}_{2k}$ is convex, then $k \leq 2$.*

Proof. Since P is convex, it intersects each of the coordinate axes in at most two points, unless it is tangent to one of the coordinate axes, and there are three consecutive collinear vertices on that axis. However this latter possibility would contradict the alternating property of P . So the only alternative is the former, in which case we have $k \leq 2$. Observe that the given inequality on k cannot be improved. \square

The following properties are easy to verify:

1. $A(\mathbf{x}, \mathbf{y}, \sigma)$ has $2k$ distinct vertices.
2. $A(\mathbf{x}, \mathbf{y}, \sigma)$ is weakly scalene.
3. The pop operation applied to the vertex p_i of $A(\mathbf{x}, \mathbf{y}, \sigma)$, ($1 \leq i \leq 2k$), yields $A(\mathbf{x}, \mathbf{y}, \sigma')$, where σ' differs from σ only in the i th bit. That is, the absolute value of the non-zero

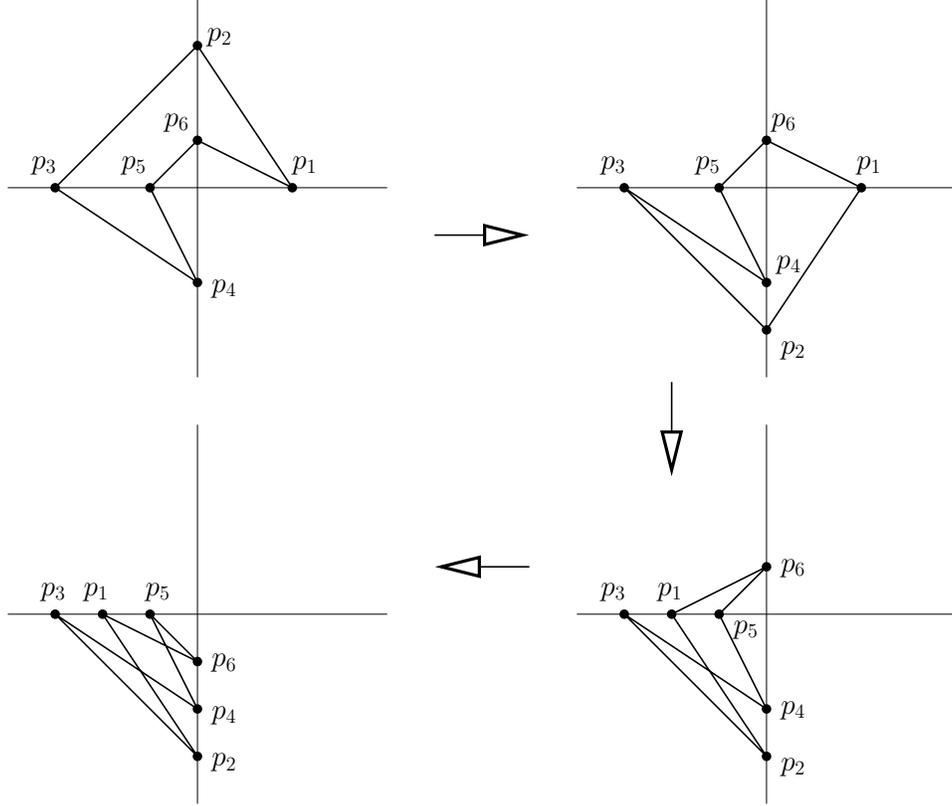


Figure 2: Sequence of three pops executed on vertices p_2 , p_1 , and p_6 of $\{p_1, p_2, p_3, p_4, p_5, p_6\}$. The corresponding polygon sequence is $A((2, 3, 1), (3, 2, 1), (+1, +1, -1, -1, -1, +1)) \Rightarrow A((2, 3, 1), (3, 2, 1), (+1, -1, -1, -1, -1, +1)) \Rightarrow A((2, 3, 1), (3, 2, 1), (-1, -1, -1, -1, -1, +1)) \Rightarrow A((2, 3, 1), (3, 2, 1), (-1, -1, -1, -1, -1, -1))$.

coordinate of p_i remains the same, with the point switching to its mirror image with respect to the origin of the axes. In particular, this implies that the family \mathcal{A}_{2k} is closed with respect to pop operations.

4. Let \mathbf{x}, \mathbf{y} be fixed, with the above properties, and σ, σ' be two sign vectors. Consider $P = A(\mathbf{x}, \mathbf{y}, \sigma)$, and $P' = A(\mathbf{x}, \mathbf{y}, \sigma')$. Then P' can be obtained from P by executing at most n pops, via: For $i = 1$ to n do: if $\sigma_i \neq \sigma'_i$, then pop p_i to p'_i .

We are now ready to prove our main result:

Theorem 1. *Let $n = 2k$, where $k \geq 3$. Any polygon in the family \mathcal{A}_n is non-convexifiable by pop operations.*

Proof. Consider a polygon $P \in \mathcal{A}_{2k}$. (We can choose P simple, or self-intersecting, as desired.) By Lemma 1, P is not convex. Apply any finite sequence of pop operations. By the second property (2.) above, the resulting polygon also belongs to \mathcal{A}_{2k} , and is therefore not convex. \square

3 Conclusion

We have shown that there exists a family of polygons that cannot be convexified by a finite sequence of pops. However, there exist many polygons that can be convexified in this way. We conclude with two questions:

1. What is the computational complexity of deciding whether a given (simple or self-crossing) polygon can be convexified by a finite sequence of pops?
2. How hard is it to find a shortest sequence of pops that convexifies a given polygon (assuming it is convexifiable in this way)? Do good approximation algorithms exist for this problem?

References

- [1] H.-K. Ahn, P. Bose, J. Czyzowicz, N. Hanusse, E. Kranakis, and P. Morin: Flipping your lid, *Geoinformatics*, **10(2)** (2000), 57–63.
- [2] O. Aichholzer, E. D. Demaine, J. Erickson, F. Hurtado, M. Overmars, M. Soss, G. Toussaint: Reconfiguring convex polygons, *Computational Geometry: Theory and Applications*, **20(1-2)** (2001), 85–95.
- [3] O. Aichholzer, C. Cortés, E. D. Demaine, V. Dujmović, J. Erickson, H. Meijer, M. Overmars, B. Balop, S. Ramaswami, and G. Toussaint: Flipturning polygons, *Discrete & Computational Geometry*, **28** (2002), 231–253.
- [4] G. Aloupis, B. Ballinger, P. Bose, M. Damian, E. D. Demaine, M. L. Demaine, R. Flatland, F. Hurtado, S. Langerman, J. O’Rourke, P. Taslakian, and G. Toussaint: Vertex pops and popturns, *Proceedings of the 19th Canadian Conference on Computational Geometry*, (CCCG 2007), Ottawa, pp. 137–140.
- [5] B. Ballinger: *Length-Preserving Transformations on Polygons*, PhD Thesis, University of California, Davis, 2003.
- [6] T. Biedl: Polygons needing many flipturns, *Discrete & Computational Geometry*, **35** (2006), 131–141.
- [7] E. D. Demaine, M. L. Demaine, T. Fevens, A. Mesa, M. Soss, D. Souvaine, P. Taslakian, and G. Toussaint: Deflating the pentagon, in *Computational Geometry and Graph Theory*, Volume 4535/2008 of LNCS, pp. 56–67.
- [8] E. D. Demaine, B. Gassend, J. O’Rourke, and G. Toussaint: Polygons Flip Finitely... Right?, *Contemporary Mathematics*, **453** (2006), 231–255.
- [9] E. D. Demaine and J. O’Rourke: *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*, Cambridge Univ. Press, Cambridge, 2007.
- [10] P. Erdős, Problem number 3763, *American Mathematical Monthly*, **42(10)** (1935), 627.
- [11] T. Fevens, A. Hernandez, A. Mesa, P. Morin, M. Soss, and G. Toussaint: Simple polygons with an infinite sequence of deflations, *Beiträge zur Algebra und Geometrie*, **42** (2001), 307–311.

- [12] B. Grünbaum: How to convexify a polygon, *Geombinatorics*, **5** (1995), 24–30.
- [13] B. Grünbaum and J. Zaks: Convexification of polygons by flips and by floipturns, *Discrete Mathematics*, **241** (2001), 333–342.
- [14] T. Kaluza: Problem 2: Konvexieren von Polygonen, *Mathematische Semesterberichte*, **28** (1981), 153–154.
- [15] N.D. Kazarinoff and R.H. Bing: On the finiteness of the number of reflections that change a non-convex plane polygon into a convex one [In Russian], *Matematicheskoe Prosveshchenie*, **36** (1961), 205–207.
- [16] K. Millett: Knotting of regular polygons in 3-space, *Journal of Knot Theory and its Ramifications*, **3(3)** (1994), 263–278.
- [17] B. de Sz. Nagy: Solution to problem number 3763, *American Mathematical Monthly*, **46(3)** (1939), 176–177.
- [18] Yu.G. Reshetnyak: On a method of transforming a non-convex polygonal line into a convex one [in Russian], *Uspehi Mat. Nauk (N.S.)*, **12(3)** (1957), 189–191.
- [19] S.A. Robertson: Inflation of planes curves, *Geometry and Topology of Submanifolds*, III, World Scientific, 1991, pp. 264–275.
- [20] S.A. Robertson and B. Wegner: Full and partial inflation of plane curves, *Intuitive Geometry, Colloquia Math. Soc. János Bolyai*, vol. 63, North-Holland 1994, pp. 389–401.
- [21] G. Sallee: Stretching chords of space curves, *Geometriae Dedicata*, **2** (1973), 311–315.
- [22] J. Stratzen and J. Brooks: A chord stretching map of a convex loop is an isometry, *Geometriae Dedicata*, **41** (1992), 51–62.
- [23] G. Toussaint: The Erdős-Nagy Theorem and its Ramifications, *Computational Geometry: Theory and Applications*, **31(3)** (1999), 219–236.
- [24] B. Wegner: Partial inflation of closed polygons in the plane, *Beiträge zur Algebra und Geometrie*, **34** (1993), 77–85.
- [25] A.Ya Yusupov: A property of simply-connected non-convex polygons [in Russian], *Uchen. Zapiski Buharsk. Gos. Pedagog. Instituta*, Tashkent, 1957, pp. 101-103.