

# THE OPAQUE SQUARE

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## ABSTRACT

The problem of finding small sets that block every line passing through a unit square was first considered by Mazurkiewicz in 1916. We call such a set an *opaque set* or a *barrier* for the square. The shortest known barrier has length  $\sqrt{2} + \frac{\sqrt{6}}{2} = 2.6389\dots$ . The current best lower bound for the length of a (not necessarily connected) barrier is 2, of which the earliest record dates back to Jones in 1964. No better lower bound is known even if the barrier is restricted to lie in the square or in its close vicinity. Under a suitable locality assumption, we replace this lower bound, for barriers consisting of finitely many straight-line segments, by  $2 + 10^{-12}$ , which represents the first, albeit small, step in a long time toward finding the length of the shortest barrier. A sharper bound is obtained for interior barriers: the length of any interior barrier for the unit square, consisting of finitely many straight-line segments, is at least  $2 + 10^{-5}$ . Two of the key elements in our proofs are: (i) formulas established by Sylvester for the measure of all lines that meet two disjoint planar convex bodies, and, even more significant, (ii) a procedure for detecting lines that are witness to the invalidity of a short bogus barrier for the square.

## Categories and Subject Descriptors

G.2.3 [Discrete Mathematics]: Applications

## Keywords

Opaque square problem, measure of a set of lines.

## 1. INTRODUCTION

The problem of finding small sets that block every line passing through a unit square was first considered by Mazurkiewicz in 1916 [34]; see also [3], [22]. Let  $C$  be a convex body in the plane. Following Bagemihl [3], a set  $\Gamma$  is an *opaque set* or a *barrier* for  $C$ , if  $\Gamma$  meets all lines

that intersect  $C$ . A barrier does not need to be connected; it may consist of one or more rectifiable arcs and its parts may lie anywhere in the plane, including the exterior of  $C$ ; see [3], [5].

*What is the length of the shortest barrier for a given convex body  $C$ ?* In spite of considerable efforts, the answer to this question is not known even in the simplest instances, such as when  $C$  is a square, a disk, or an equilateral triangle; see [6], [7, Problem A30], [14], [16], [17], [20, Section 8.11], [23, Problem 12]. Some entertaining variants of the problem appeared in different forms in the literature [2, 5, 21, 25, 29, 30].

A barrier blocks any line of sight across the region  $C$  or detects any ray that passes through it. Potential applications are in guarding and surveillance [8]. Here we focus on the case when  $C$  is a square. The shortest barrier known for the unit square, of length  $2.639\dots$ , is illustrated in Figure 1 (right). It is conjectured to be optimal. The current best lower bound, 2, was first mentioned by Jones [24] in 1964<sup>1</sup>.

The type of curve barriers considered may vary: the most restricted are barriers made from single continuous arcs, then connected barriers, and lastly, arbitrary (possibly disconnected) barriers. For the unit square, the shortest known in these three categories have lengths  $3$ ,  $1 + \sqrt{3} = 2.7320\dots$  and  $\sqrt{2} + \frac{\sqrt{6}}{2} = 2.6389\dots$ , respectively. They are depicted in Figure 1. Obviously, disconnected barriers offer the greatest freedom of design. For instance, Kawohl [27] showed that the barrier in Figure 1 (right) is optimal in the class of curves with at most two components restricted to the square. For the unit disk, the shortest known barrier consists of three arcs. See also [16, 20].

Barriers can be also classified by where they can be located. In certain instances, it might be infeasible to construct barriers guarding a specific domain outside the domain, since that part might belong to others. Following [12] we call such barriers constrained to the interior and the

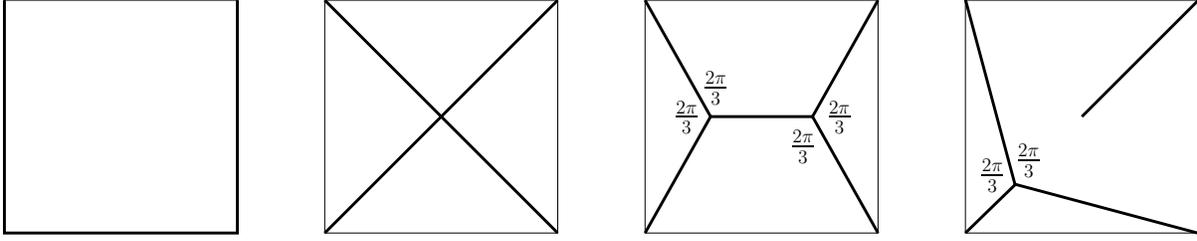
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<sup>1</sup>A note of caution for the non-expert about the subtlety of the problem: an arxiv submission [13] dated May 2010 claimed a first small improvement in the old lower bound of 2 for a unit square; its proof had a fatal error, and the submission was soon after withdrawn by the authors. Further, at least two conference submissions by two other groups of authors were made in the last 3 years claiming (erroneous) improvements in the same lower bound of 2 for a unit square; both submissions were rejected at the respective conferences and the authors were notified of the errors discovered. As a most recent development, a preprint [26] making the same claim was posted just before our paper went to print.



**Figure 1:** In the left three unit squares are three connected barriers of lengths 3,  $2\sqrt{2} = 2.8284\dots$ , and  $1 + \sqrt{3} = 2.7320\dots$ , respectively. In the rightmost unit square, the diagonal segment  $[(1/2, 1/2), (1, 1)]$  together with three segments connecting the corners  $(0, 1)$ ,  $(0, 0)$ ,  $(1, 0)$  to the point  $(\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{6})$  yield a barrier of length  $\sqrt{2} + \frac{\sqrt{6}}{2} = 2.639\dots$

boundary of the domain, *interior*. For example, all four barriers for the unit square illustrated in Figure 1 are interior barriers. On the other hand, certain instances may prohibit barriers lying in the interior of a domain. We call a barrier constrained to the exterior and the boundary of the domain, *exterior*. For example, since the first barrier from the left in Figure 1 is contained in the boundary of the square, it is also an exterior barrier.

### Early algorithms and other related work.

Two algorithms, proposed by Akman [1] and respectively Dubish [9] in the late 1980s, claiming to compute shortest interior-restricted barriers, were refuted by Shermer [38] in the early 1990s. Shermer [38] proposed a new algorithm, which shared the same fate and was refuted recently by Provan et al. [35]; see also [11]. As of today, no exact algorithm for computing a shortest (interior-restricted or unrestricted) barrier is known. Even though we have so little control on the shape or length of optimal barriers, barriers whose lengths are somewhat longer can be computed efficiently for any given convex polygon. Various approximation algorithms with a small constant ratio have been obtained recently by Dumitrescu et al. [12].

If instead of curve barriers, we want to find *discrete* barriers consisting of as few points as possible with the property that every line intersecting  $C$  gets closer than  $\varepsilon > 0$  to at least one of them in some fixed norm, we arrive at a problem raised by László Fejes Tóth [18, 19] and subsequently studied by others [4, 28, 33, 36, 40]. The problem of short barriers has attracted many other researchers and has been studied at length; see also [6, 15, 23, 31, 32].

### Our Results.

A *finite segment barrier* is a barrier consisting of finitely many straight-line segments. In this paper we consider only finite segment barriers. In Section 3, we prove:

**THEOREM 1.** *The length of any finite segment barrier for the unit square  $U$  restricted to the square of side length 2 concentric and homothetic to  $U$  is at least  $2 + 10^{-12}$ .*

The possibility that parts of the barrier may be located outside of the unit square  $U$  only adds to the difficulty of obtaining a good lower bound. Indeed, for the special case of barriers whose location is restricted to  $U$ , the proof of the inequality in Theorem 1 becomes slightly easier. Moreover, a better lower bound can be obtained (by an analytical proof along the same lines) in this case; however we omit this

exercise. We then go one step further, and by combining the methods developed in proving Theorem 1 with the use of linear programming, in Section 4 we establish a sharper bound:

**THEOREM 2.** *The length of any interior finite segment barrier for a unit square is at least  $2 + 10^{-5}$ .*

Due to space limitations, the proofs of Lemmas 7 and 9 are omitted from this extended abstract. All missing proofs can be found in the arxiv version [10].

## 2. PRELIMINARIES

For a curve  $\gamma$ , let  $|\gamma|$  denote the length of  $\gamma$ . Similarly, if  $\Gamma$  is a set of curves, let  $|\Gamma|$  denote the total length of the curves in  $\Gamma$ .

We first review three different proofs for the lower bound of 2 (the current best lower bound for the unit square).

*First proof:* The first proof, Lemma 1, is general and applies to any planar convex body; its proof is folklore, based on Cauchy's integral formula [12, 17, 24, 37]. Let  $\Gamma = \{s_1, \dots, s_n\}$  consist of  $n$  segments of lengths  $\ell_i = |s_i|$ , where  $L = |\Gamma| = \sum_{i=1}^n \ell_i$ . Let  $\alpha_i \in [0, \pi)$  be the angle made by  $s_i$  with the  $x$ -axis. For each direction  $\alpha \in [0, \pi)$ , the blocking (opaqueness) condition for a convex body  $C$  requires

$$\sum_{i=1}^n \ell_i |\cos(\alpha - \alpha_i)| \geq w(\alpha). \quad (1)$$

Here  $w(\alpha)$  is the width of  $C$  in direction  $\alpha$ , i.e., the minimum width of a strip of parallel lines enclosing  $C$ , whose lines are orthogonal to direction  $\alpha$ . Integrating this inequality over the interval  $[0, \pi]$  yields the following.

**LEMMA 1.** *Let  $C$  be a convex body in the plane and let  $\Gamma$  be a barrier for  $C$ . Then the length of  $\Gamma$  is at least  $\frac{1}{2} \cdot \text{per}(C)$ .*

For the unit square we have  $\text{per}(U) = 4$  thus Lemma 1 yields the lower bound  $L \geq 2$ .

*Second proof:* We make use of formulas established by Sylvester [39]; see also [37, pp. 32–34]. The setup is as follows. For a planar convex body  $K$ , the measure of all lines that meet  $K$  is equal to  $\text{per}(K)$ . In particular, if  $K$  degenerates to a segment  $s$ , the measure of all lines that meet  $s$  is equal to  $2|s|$ .

Let  $\mathcal{G}$  denote all lines that meet  $U$ ; let  $\mathcal{G}_i$  denote all lines that meet a segment  $s_i \in \Gamma$ . The measure of all lines that

meet  $U$  is equal to  $m(\mathcal{G}) = \text{per}(U) = 4$ . Since  $m(\cdot)$  is a measure, we have

$$4 = m(\mathcal{G}) \leq \sum_{i=1}^n m(\mathcal{G}_i) = 2 \sum_{s_i \in \Gamma} |s_i| = 2L.$$

It follows that  $2L \geq 4$  or  $L \geq 2$ , as required.

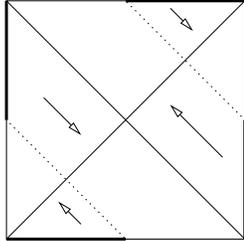
*Third proof* (due to O. Ozkan; reported in [8]): This proof is specific to the square. Let  $\Gamma = \{s_1, \dots, s_n\}$  consist of  $n$  segments of lengths  $\ell_i = |s_i|$ , where  $L = |\Gamma| = \sum_{i=1}^n \ell_i$ . Let  $d_1$  and  $d_2$  be the two diagonals of  $U$ . Let  $\theta_i \in [0, \pi)$  be the angle made by  $s_i$  with the first diagonal  $d_1$ . Consider the blocking (opaqueness) conditions only for these two directions, that is, for  $\alpha = \pi/4$ , and  $\alpha = 3\pi/4$ . Equation (1) for these two directions gives now:

$$\sum_{i=1}^n \ell_i |\cos \theta_i| \geq \sqrt{2}, \quad \text{and} \quad \sum_{i=1}^n \ell_i |\sin \theta_i| \geq \sqrt{2}. \quad (2)$$

Consequently, since  $|\cos \theta_i| + |\sin \theta_i| \leq \sqrt{2}$  holds for any angle  $\theta_i$ , we have

$$\begin{aligned} 2\sqrt{2} &\leq \sum_{i=1}^n \ell_i (|\cos \theta_i| + |\sin \theta_i|) \leq \sqrt{2} \sum_{i=1}^n \ell_i \\ \Rightarrow \quad L &= \sum_{i=1}^n \ell_i \geq 2. \end{aligned} \quad (3)$$

An obvious question is whether any of these proofs can give more. Regarding the third proof, if one considers only those four main directions used there, namely the two coordinate axes and the two diagonal directions, there is no hope left. Interestingly enough, there exists a structure (imperfect barrier) of length 2 made of four axis-parallel segments, that perfectly blocks (i.e., with no overlap) the four main directions; see Figure 2. Thus one needs to find other directions that are not opaque besides these four. This observation was the starting point of our investigations.



**Figure 2:** This structure of length 2 perfectly blocks the four main directions; here shown for the main diagonal.

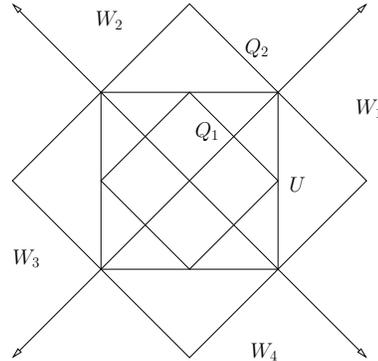
*Setup for the new lower bound of  $2 + 10^{-12}$ .*

We set four parameters:

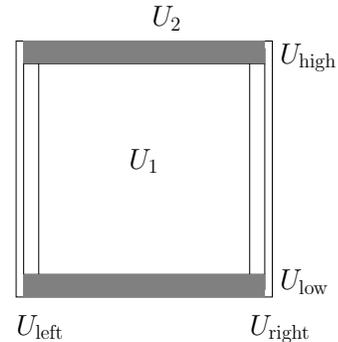
- $\delta = 10^{-12}$ ,  $\phi = \arcsin 10^{-4}$ ; note that  $10^8 \delta = \sin \phi$ .
- $w_1 = 1/20$ ,  $w_2 = 1/1000$ .

Refer now to Figures 3 and 4 which illustrate various regions we define below in relation to the unit square  $U$  (recall that parts of the barrier may be located in the exterior of  $U$ ).

- $U = [0, 1]^2$  is an (axis-aligned) unit square centered at point  $o = (1/2, 1/2)$ .
- $U_1 = [w_1, 1-w_1]^2$  is an (axis-aligned) square concentric with  $U$ .
- $U_2 = [-w_2, 1+w_2]^2$  is an (axis-aligned) square concentric with  $U$ .
- $U_3 = [-1/2, 3/2]^2$  is an (axis-aligned) square of side length 2 concentric with  $U$ .
- $Q_1$  is a square of side length  $\sqrt{2}/2$  concentric with  $U$  and rotated by  $\pi/4$ .
- $Q_2$  is a square of side length  $\sqrt{2}$  concentric with  $U$  and rotated by  $\pi/4$ .
- $V = [0, 1] \times (-\infty, +\infty)$  is the (infinite) vertical strip of unit width containing  $U$ .
- $H = (-\infty, +\infty) \times [0, 1]$  is the (infinite) horizontal strip of unit width containing  $U$ .
- $d_1$  is  $U$ 's diagonal of positive slope and  $d_2$  is  $U$ 's diagonal of negative slope.
- $W_1, W_2, W_3, W_4$  are the four wedges centered at  $o$  and bounded by the lines supporting the two diagonals of  $U$ , and directed to the right, up, left, and down (i.e., in counterclockwise order).



**Figure 3:** The unit square  $U$  and the rotated squares  $Q_1$  and  $Q_2$ .



**Figure 4:**  $U_1$  and  $U_2$ ; two thin rectangles  $U_{\text{low}}, U_{\text{high}} \subset U_2 \setminus U_1$  (out of the four) are shaded.

- ◇  $U_{\text{right}} = [1 - w_1, 1 + w_2] \times [0, 1]$  is a thin rectangle of width  $w_1 + w_2$  and height 1 whose right side coincides with the right side of  $U_2$ . Similarly, denote by  $U_{\text{left}}, U_{\text{low}}, U_{\text{high}}$ , the analogous rectangles contained in  $U_2 \setminus U_1$  and sharing the corresponding sides of  $U_2$ , as indicated.

Observe that  $U_1 \subset U \subset U_2$ , and  $Q_1 \subset U \subset Q_2$ . Note also that the inclusion  $U_1 \subset U_2 \setminus (U_{\text{low}} \cup U_{\text{high}} \cup U_{\text{left}} \cup U_{\text{right}})$  is strict. Complementary regions such as  $\mathbb{R}^2 \setminus U_2, \mathbb{R}^2 \setminus Q_2, \mathbb{R}^2 \setminus V, \mathbb{R}^2 \setminus H$ , are denoted by  $\overline{U_2}, \overline{Q_2}, \overline{V}, \overline{H}$ .

We say that a segment (or a line) is *almost horizontal* if its direction angle belongs to the interval  $[-\phi, \phi]$ . Similarly, we say that a segment (or a line) is *almost vertical* if its direction angle belongs to the interval  $[\frac{\pi}{2} - \phi, \frac{\pi}{2} + \phi]$ . Let  $\Gamma$  be a segment barrier for  $U$  of length  $L = |\Gamma|$ . Let  $X$  be the set of almost horizontal segments in  $\Gamma$ , and  $Y$  be the set of almost vertical segments in  $\Gamma$ . Let  $Z$  be the rest of the segments in  $\Gamma$ . Clearly, we have  $L = |\Gamma| = |X| + |Y| + |Z|$ .

### 3. LOCAL BARRIERS: PROOF OF THEOREM 1

Let  $\Gamma$  be a finite segment barrier for  $U$  of length  $L = |\Gamma| \geq 2$ . Assume for contradiction that  $L \leq 2 + \delta$ . We first establish several structural properties of  $\Gamma$ :

- $\Gamma$  must consist mostly of almost horizontal segments and almost vertical segments. The total length of the almost horizontal segments must be close to 1, and similarly, the total length of the almost vertical segments must be close to 1.
- The total length of the segments in the exterior of  $U_2$  must be small.
- For each side  $s$  of  $U$ , a thin rectangle parallel to  $s$  and enclosing  $s$  must contain a set of significant weight consisting of barrier segments almost parallel to  $s$ .

Once established, these structural properties of  $\Gamma$  are used to find a line that is witness to the invalidity of  $\Gamma$ . By way of contradiction, the lower bound in Theorem 1 will consequently follow. Let us record our initial assumptions to start with:

$$2 \leq L = |X| + |Y| + |Z| \leq 2 + \delta. \quad (4)$$

To begin our proof, we first refine Ozkan's argument (in Section 2) for the lower bound of 2. We first show that the total length of the segments in  $Z$  is small.

LEMMA 2. *The total length of the segments in  $Z$  satisfies:  $|Z| \leq 2 \cdot 10^8 \delta = 2 \sin \phi$ .*

**Proof.** Put  $c = 2 \cdot 10^8$ , and assume for contradiction that  $|Z| \geq c\delta$ , hence  $|X| + |Y| \leq 2 + \delta - c\delta = 2 - (c - 1)\delta$ . Observe that for any segment in  $Z$ , we have (with  $\theta_i$  as in the respective proof)

$$|\cos \theta_i| + |\sin \theta_i| \leq \cos\left(\frac{\pi}{4} - \phi\right) + \sin\left(\frac{\pi}{4} - \phi\right) := a.$$

Note that

$$\begin{aligned} a &= \frac{\sqrt{2}}{2} \cos \phi + \frac{\sqrt{2}}{2} \sin \phi + \frac{\sqrt{2}}{2} \cos \phi - \frac{\sqrt{2}}{2} \sin \phi \\ &= \sqrt{2} \cos \phi < \sqrt{2}. \end{aligned}$$

By the assumption, the first inequality in (3) yields

$$2\sqrt{2} \leq (2 - (c - 1)\delta)\sqrt{2} + c\delta < 2\sqrt{2},$$

a contradiction. Indeed, the second inequality in the above chain is equivalent to

$$\cos \phi < \frac{c - 1}{c},$$

which holds since

$$\cos \phi = \sqrt{1 - \sin^2 \phi} = \sqrt{1 - 10^{-8}} < 1 - \frac{1}{2 \cdot 10^8} = \frac{c - 1}{c}. \quad \square$$

Next we show that the total length of the almost horizontal segments is close to 1, and similarly, that the total length of the almost vertical segments is close to 1.

LEMMA 3. *The following inequalities hold:*

$$\begin{aligned} 1 - \frac{7}{2} \sin \phi &\leq |X| \leq 1 + \frac{3}{2} \sin \phi \\ 1 - \frac{7}{2} \sin \phi &\leq |Y| \leq 1 + \frac{3}{2} \sin \phi. \end{aligned} \quad (5)$$

**Proof.** It follows from (4) that  $|X| + |Z| \leq 2 + \delta - |Y|$ . We first prove the upper bounds:  $|X| \leq 1 + \frac{3}{2} \sin \phi$  and  $|Y| \leq 1 + \frac{3}{2} \sin \phi$ . Assume first for contradiction that  $|Y| \geq 1 + \frac{3}{2} \sin \phi$ . The total projection length of the segments in  $\Gamma$  on the  $x$ -axis is at most

$$\begin{aligned} |X| + |Z| + |Y| \cos\left(\frac{\pi}{2} - \phi\right) &\leq (2 + \delta - |Y|) + |Y| \sin \phi \\ &= 2 + \delta - |Y|(1 - \sin \phi) \\ &\leq 2 + \delta - \left(1 + \frac{3}{2} \sin \phi\right)(1 - \sin \phi) \\ &= 1 + \delta - \frac{1}{2} \sin \phi + \frac{3}{2} \sin^2 \phi < 1, \end{aligned}$$

i.e., smaller than the corresponding unit width of  $U$ . This contradicts the opaqueness condition for vertical rays, hence  $|Y| \leq 1 + \frac{3}{2} \sin \phi$ . Similarly we establish that  $|X| \leq 1 + \frac{3}{2} \sin \phi$ .

We now prove the lower bounds:  $|X| \geq 1 - \frac{7}{2} \sin \phi$  and  $|Y| \geq 1 - \frac{7}{2} \sin \phi$ . Assume for contradiction that  $|X| \leq 1 - \frac{7}{2} \sin \phi$ . Using the previous upper bound on  $|Y|$ , the total projection length of the segments in  $\Gamma$  on the  $x$ -axis is

$$\begin{aligned} |X| + |Z| + |Y| \sin \phi &\leq \left(1 - \frac{7}{2} \sin \phi\right) \\ &\quad + 2 \sin \phi + \left(1 + \frac{3}{2} \sin \phi\right) \sin \phi \\ &= 1 - \frac{1}{2} \sin \phi + \frac{3}{2} \sin^2 \phi < 1, \end{aligned}$$

i.e., smaller than the corresponding width of  $U$ . This is a contradiction, hence  $|X| \geq 1 - \frac{7}{2} \sin \phi$ . Similarly we establish that  $|Y| \geq 1 - \frac{7}{2} \sin \phi$ .  $\square$

Further restrictions on the placement of the segments in  $\Gamma$  are given by the following two lemmas.

LEMMA 4. *The following inequalities hold:*

$$|X \cap \overline{V}| \leq \frac{9}{2} \sin \phi \quad \text{and} \quad |Y \cap \overline{H}| \leq \frac{9}{2} \sin \phi. \quad (6)$$

**Proof.** By Lemma 3, we have  $|Y| \geq 1 - \frac{7}{2} \sin \phi$ . Assume for contradiction that  $|X \cap \overline{V}| \geq \frac{9}{2} \sin \phi$ . It follows that the total projection length of the segments in  $\Gamma$  on the interval  $[0, 1]$  of the  $x$ -axis is at most

$$\begin{aligned} |X \cap V| + |Z| + |Y| \sin \phi &= |X| - |X \cap \overline{V}| + |Z| + |Y| \sin \phi \\ &\leq 2 + \delta - \frac{9}{2} \sin \phi - |Y| + |Y| \sin \phi \\ &= 2 + \delta - \frac{9}{2} \sin \phi - |Y|(1 - \sin \phi) \\ &\leq 2 + \delta - \frac{9}{2} \sin \phi - \left(1 - \frac{7}{2} \sin \phi\right)(1 - \sin \phi) \\ &\leq 1 + \delta - \frac{7}{2} \sin^2 \phi < 1, \end{aligned}$$

i.e., smaller than the corresponding unit width. This is a contradiction, hence  $|X \cap \overline{V}| \leq \frac{9}{2} \sin \phi$ . Similarly we establish that  $|Y \cap \overline{H}| \leq \frac{9}{2} \sin \phi$ .  $\square$

LEMMA 5. *Let  $I, J \subset \mathbb{R}$  be two intervals (not necessarily contained in  $[0, 1]$ ). Then*

$$\begin{aligned} |X \cap I \times (-\infty, \infty)| &\leq |I \cap [0, 1]| + 5 \sin \phi, \text{ and similarly} \\ |Y \cap (-\infty, \infty) \times J| &\leq |J \cap [0, 1]| + 5 \sin \phi. \end{aligned}$$

**Proof.** Put  $\overline{I} = [0, 1] \setminus I$ , and  $\overline{J} = [0, 1] \setminus J$ . Assume for contradiction that  $|X \cap I \times (-\infty, \infty)| \geq |I \cap [0, 1]| + 5 \sin \phi$ . Then by Lemma 4 we have

$$\begin{aligned} |X \cap \overline{I} \times (-\infty, \infty)| &= |X| - |X \cap I \times (-\infty, \infty)| \\ &\leq \left(1 + \frac{3}{2} \sin \phi\right) - |I \cap [0, 1]| - 5 \sin \phi \\ &= 1 - |I \cap [0, 1]| - \frac{7}{2} \sin \phi. \end{aligned}$$

However, since

$$\begin{aligned} &|X \cap \overline{I} \times (-\infty, \infty)| + |Z| + |Y| \sin \phi \\ &\leq \left(1 - |I \cap [0, 1]| - \frac{7}{2} \sin \phi\right) + 2 \sin \phi + \left(1 + \frac{3}{2} \sin \phi\right) \sin \phi \\ &= 1 - |I \cap [0, 1]| - \frac{1}{2} \sin \phi + \frac{3}{2} \sin^2 \phi < 1 - |I \cap [0, 1]|, \end{aligned}$$

the vertical lines intersecting the lower side of  $U$  in  $\overline{I}$  are not blocked, which is a contradiction.

The proof of the second inequality is analogous.  $\square$

Next we show that the total length of the segments in  $\Gamma$  lying in the exterior of  $Q_2$  is small.

LEMMA 6. *The following inequality holds:*

$$|\Gamma \cap \overline{Q_2}| \leq 4\delta. \quad (7)$$

**Proof.** Assume for contradiction that  $|\Gamma \cap \overline{Q_2}| \geq 4\delta$ . Observe that any segment in  $\Gamma \cap \overline{Q_2}$  projects either in the exterior of  $d_1$  on its supporting line, or in the exterior of  $d_2$  on its supporting line. It follows from (4) that the total length of the segments in  $\Gamma$  that project (at least in part) on both diagonals is at most  $2 + \delta - 4\delta = 2 - 3\delta$ . Therefore the total projection length of the segments in  $\Gamma$  on the two diagonals (see also (3)) is at most

$$(2 - 3\delta)\sqrt{2} + 4\delta = 2\sqrt{2} - (3\sqrt{2} - 4)\delta < 2\sqrt{2},$$

that is, smaller than the sum of the lengths of the two diagonals. This is a contradiction, hence  $|\Gamma \cap \overline{Q_2}| \leq 4\delta$ .  $\square$

Next we show that the total length of the segments in  $\Gamma$  lying in the exterior of  $U_2$  is small. We use again formulas established by Sylvester [39]; see also [37, pp. 32–34] and the second proof for the bound of 2 in Section 2. For a planar convex body  $K$ , the measure of all lines that meet  $K$  is equal to  $\text{per}(K)$ . In particular, the measure of all lines that meet a segment  $s$  is equal to  $2|s|$ . Let now  $K_1, K_2$  be two disjoint planar convex bodies and let  $L_1$  and  $L_2$  be the lengths of the boundaries  $\partial K_1, \partial K_2$ . The *external cover*  $C_{\text{ext}}$  of  $K_1$  and  $K_2$  is the boundary of  $\text{conv}(K_1 \cup K_2)$ . The external cover may be interpreted as a closed elastic string drawn about  $K_1$  and  $K_2$ . Let  $L_{\text{ext}}$  denote the length of  $C_{\text{ext}}$ . The *internal cover*  $C_{\text{int}}$  of  $K_1$  and  $K_2$  is the closed curve realized by a closed elastic string drawn about  $K_1$  and  $K_2$  and crossing over at a point between  $K_1$  and  $K_2$ . Let  $L_{\text{int}}$  denote the length of  $C_{\text{int}}$ . Then, according to [39], the measure of all lines that meet  $K_1$  and  $K_2$  is  $L_{\text{int}} - L_{\text{ext}}$ . We need a technical lemma:

LEMMA 7. *Let  $B$  be a convex body and let  $s$  be a segment disjoint from  $B$ . Let  $\theta$  be the maximum angle of a minimum cone  $C$  that contains  $B$  and has apex  $c$  in  $s$ , that is,  $\theta = \max_{c \in s} \min_{C \supseteq B} \angle C$ . Then the measure of all lines that meet both  $B$  and  $s$  is at most  $2 \sin \frac{\theta}{2} \cdot |s|$ .*

COROLLARY 1. *Consider a segment  $s \in \Gamma \cap \overline{U_2}$ . Then the measure of all lines that meet  $s$  and  $U$  is at most  $(1/4 + 10^{-6})^{-1/2} |s| < 2|s|$ .*

**Proof.** Consider the setup in Lemma 7, with  $B = U$ . For  $s \subset \overline{U_2}$ ,  $\theta/2 < \pi/2$  is maximized when the apex of the cone anchored in  $s$  is the midpoint of one of the sides of  $U_2$ . In this case we have

$$\sin \frac{\theta}{2} = \frac{1}{2\sqrt{\frac{1}{4} + w_2^2}} = \frac{1}{2\sqrt{\frac{1}{4} + 10^{-6}}}.$$

It follows from Lemma 7 that the measure of all lines that meet  $s$  and  $U$  is at most  $2 \sin \frac{\theta}{2} \cdot |s| = (1/4 + 10^{-6})^{-1/2} |s|$ , as required.  $\square$

LEMMA 8. *The following inequality holds:*

$$|\Gamma \cap \overline{U_2}| \leq (5 \cdot 10^5 + 2) \delta = \frac{1}{200} \sin \phi + 2\delta. \quad (8)$$

**Proof.** Let  $\mathcal{G}$  denote all lines that meet  $U$ ; let  $\mathcal{G}_2$  denote all lines that meet some segment in  $\Gamma \cap U_2$ , and let  $\mathcal{G}_{\overline{2}}$  denote all lines that meet  $U$  and some segment in  $\Gamma \cap \overline{U_2}$ .

The measure of all lines that meet  $U$  is equal to  $m(\mathcal{G}) = \text{per}(U) = 4$ . Since  $m(\cdot)$  is a measure, we have

$$\begin{aligned} 4 &= m(\mathcal{G}) \leq m(\mathcal{G}_2) + m(\mathcal{G}_{\overline{2}}) \\ &\leq \sum_{s_i \in \Gamma \cap U_2} 2|s_i| + \sum_{s_i \in \Gamma \cap \overline{U_2}} (1/4 + 10^{-6})^{-1/2} |s_i| \\ &= 2 \sum_{s_i \in \Gamma} |s_i| - \sum_{s_i \in \Gamma \cap \overline{U_2}} (2 - (1/4 + 10^{-6})^{-1/2}) |s_i| \\ &\leq 2(2 + \delta) - \sum_{s_i \in \Gamma \cap \overline{U_2}} (2 - (1/4 + 10^{-6})^{-1/2}) |s_i|. \end{aligned}$$

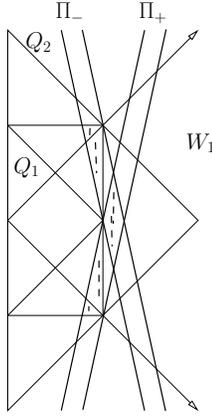
For the second inequality above we have used Corollary 1.

It follows that

$$\begin{aligned} |\Gamma \cap \overline{U_2}| &= \sum_{s_i \in \Gamma \cap \overline{U_2}} |s_i| \leq \frac{2}{2 - (1/4 + 10^{-6})^{-1/2}} \delta \\ &\leq (5 \cdot 10^5 + 2) \delta. \end{aligned}$$

□

Refer to Figure 5. Let  $\ell'_+$  denote the line incident to the points  $(19/20, 0)$  and  $(1, 1/2)$ ; let  $\ell''_+$  denote the parallel line incident to the point  $(1, 0)$ . Let  $\Pi_+$  denote the parallel strip bounded by  $\ell'_+$  and  $\ell''_+$ . Let  $\Pi_-$  denote the parallel strip obtained by reflecting  $\Pi_+$  about the horizontal line  $y = 1/2$ . Recall that  $w_1 = 1/20$ . Observe that the slope of  $\ell'_+$  is  $\frac{1/2}{w_1} = \frac{1}{2w_1} = 10$ , and let  $\alpha$  denote the angle made by  $\ell'_+$  with a vertical line. Observe that  $\tan \alpha = 2w_1 = 1/10 \gg \tan \phi \approx 10^{-4}$ .



**Figure 5: Opaqueness with respect to two parallel strips  $\Pi_+$  and  $\Pi_-$  (bounded by bold lines) covering the right side of  $U$ .**

LEMMA 9. *The following inequalities hold:*

$$\begin{aligned} |X \cap U_{\text{low}}| &\geq 0.45. \\ |X \cap U_{\text{high}}| &\geq 0.45. \\ |Y \cap U_{\text{left}}| &\geq 0.45. \\ |Y \cap U_{\text{right}}| &\geq 0.45. \end{aligned}$$

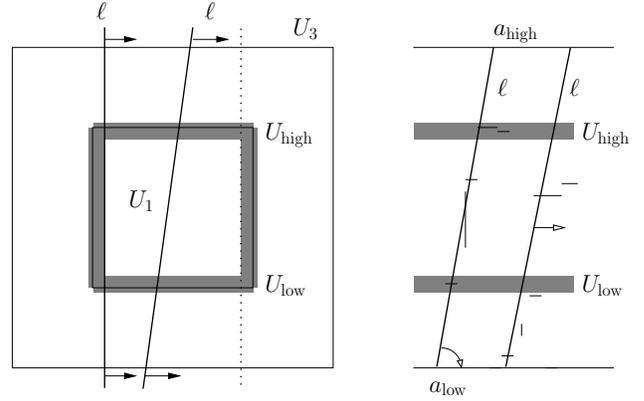
### Last step in the proof.

We describe a procedure ADVANCE for finding a line that is witness to the invalidity of  $\Gamma$  with respect to  $U$ . For convenience we refer to such an event as terminating the procedure with *success*. According to our assumption for  $\Gamma$  being a barrier, if this happens, it will be an obvious contradiction. The analysis of ADVANCE employs a potential argument ultimately based on the inequalities established in Lemma 9. The fact that  $\Gamma$  lies in  $U_3$  is key to the analysis of ADVANCE, and thus to our proof.

The initial position of the sweep-line  $\ell$  is the vertical line  $x = w_1$ . While  $\ell$  is infinite, it is convenient to view it as anchored at its intersection points with the two horizontal sides of  $U_3$ :  $a_{\text{low}}$  on the lower side and  $a_{\text{high}}$  on the higher side; the two anchor points change as  $\ell$  changes its position. The

line  $\ell$  moves right across the central part of  $U$  (resp.,  $U_2$ ), in the sense that its anchor points are stationary or move to the right on the corresponding sides of  $U_3$ , as follows. See fig. 6.

1. If  $\ell$  intersects segments in  $X \cap U_{\text{high}}$ , then  $\ell$  rotates clockwise around  $a_{\text{low}}$  until this condition fails. If  $\ell$  intersects segments in  $X \cap U_{\text{low}}$ , then  $\ell$  rotates counterclockwise around  $a_{\text{high}}$  until this condition fails. (If  $\ell$  does not intersect segments in  $X \cap U_{\text{low}}$  or  $X \cap U_{\text{high}}$ , rule 2 applies.)
2. If  $\ell$  intersects other segments of  $\Gamma$ , then  $\ell$  moves right remaining parallel to itself until this condition fails. The two anchor points  $a_{\text{low}}$  and  $a_{\text{high}}$  move right by the same amount on the corresponding sides of  $U_3$ .



**Figure 6: Left: the sweep-line in procedure ADVANCE moving right across the central part of  $U$  (the central subrectangles of  $U_{\text{low}}, U_{\text{high}} \subset U_2 \setminus U_1$ ). Right: two cases (rotation and translation) of charges with the sweep-line.**

We next show that ADVANCE achieves success before any of its two anchor points reaches the vertical line  $x = 1 - w_1$  (supporting the left side of  $U_{\text{right}}$ ).

LEMMA 10. *The following properties hold:*

- (i) *During the execution of ADVANCE, the slope of  $\ell$  in absolute value is at least*

$$\tan \beta = \frac{3/2 - w_1 - x_1 \sin \phi}{x_1 \cos \phi} \geq 2.635,$$

where  $x_1 = |X| - 0.45$ .

- (ii) *The total advance of the higher anchor point caused by rotations of  $\ell$  (sweeping over segments in  $X \cap U_{\text{high}}$ ) around the lower anchor point is at most  $x_3 = \frac{2}{\tan \beta} \leq 0.76$ .*

- (iii) *The total advance of an anchor point caused by translations of  $\ell$  over segments in  $X$  is at most*

$$\frac{\sin(\beta + \phi)}{\sin \beta} x_4 \leq 1.01 x_4,$$

where  $x_4 = |X| - |X \cap U_{\text{low}}| - |X \cap U_{\text{high}}|$ .

(iv) The total advance of an anchor point caused by translations of  $\ell$  over segments in  $Y$  is at most

$$\frac{\cos(\beta - \phi)}{\sin \beta} y_1 \leq 0.38 y_1,$$

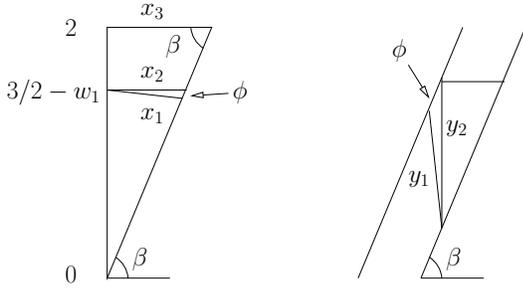
where  $y_1 = |Y| - |Y \cap U_{\text{left}}| - |Y \cap U_{\text{right}}|$ .

(v) The total advance of an anchor point caused by translations of  $\ell$  over segments in  $Z$  is at most

$$\frac{|Z|}{\sin \beta} \leq 1.1 |Z|.$$

**Proof.** For simplicity, throughout this proof, we denote a segment and its length by the same letter when there is no danger of confusion.

(i) Observe that the slope of  $\ell$  can change only during rotation. Each rotation is attributed to some segment in  $X \cap U_{\text{high}}$  or to some segment in  $X \cap U_{\text{low}}$ . We make the argument for the first case; during rotation the anchor point is fixed on the lower side of  $U_3$ . Since the initial position of  $\ell$  is vertical, the slope of  $\ell$  cannot exceed the slope of the hypotenuse of the right triangle in Figure 7, where all segments in  $X \cap U_{\text{high}}$  have been concatenated and made collinear in the segment  $x_1$  which makes an angle of  $\phi$  with a horizontal line. We now determine the slope of the hypotenuse. Let  $x_2$  be the



**Figure 7: Left: bounding the advance of the higher anchor point caused by rotations over segments in  $X \cap U_{\text{high}}$ . Right: bounding the advance of both anchor points caused by translations over segments in  $Y$ .**

horizontal segment incident to the higher endpoint of  $x_1$ . By Lemma 3,  $x_1 \leq 1 + 1.5 \sin \phi - 0.45 = 0.55 + \sin \phi$ . By the law of sines in the small triangle with sides  $x_1$  and  $x_2$ , we have

$$\frac{x_1}{\sin \beta} = \frac{x_2}{\sin(\beta + \phi)} \Rightarrow x_2 = \left( \cos \phi + \frac{\sin \phi}{\tan \beta} \right) x_1. \quad (9)$$

We also have

$$\tan \beta = \frac{3/2 - w_1}{x_2} \Rightarrow x_2 = \frac{3/2 - w_1}{\tan \beta}. \quad (10)$$

Putting (9) and (10) together yields

$$\begin{aligned} \tan \beta &= \frac{3/2 - w_1 - x_1 \sin \phi}{x_1 \cos \phi} \\ &\geq \frac{1.5 - w_1}{(0.55 + 1.5 \sin \phi) \cos \phi} - \tan \phi \geq 2.635. \end{aligned} \quad (11)$$

(ii) Refer to Figure 7 (left). Clearly, the total advance of the higher anchor point is at most  $x_3 = \frac{2}{\tan \beta} \leq 0.76$ .

(iii) As in (i), the maximum advance is achieved when the slope of  $\ell$  is the smallest in absolute value and all segments in  $X$  make an angle of  $\phi$  clockwise below the horizontal line. The total length of segments in  $X$  contributing to translations of  $\ell$  is clearly bounded from above by  $x_4 = |X| - |X \cap U_{\text{low}}| - |X \cap U_{\text{high}}|$ . As in (9), the total advance of each anchor point is at most  $\frac{\sin(\beta + \phi)}{\sin \beta} x_4 \leq 1.01 x_4$ .

(iv) Refer to Figure 7 (right). The maximum advance is achieved when the slope of  $\ell$  is the smallest in absolute value. We can assume that all segments swept over are collinear in a segment  $y_1$  that makes an angle of  $\phi$  with the vertical direction, as shown in the figure; here  $y_2$  is a vertical segment sharing an endpoint with  $y_1$ . By the law of sines in the small triangle with sides  $y_1$  and  $y_2$ , we have

$$\begin{aligned} \frac{y_1}{\sin(\pi/2 - \beta)} &= \frac{y_2}{\sin(\phi + \pi/2 - \beta)} \Rightarrow \\ y_2 &= \frac{\cos(\beta - \phi)}{\sin \beta} y_1 \leq 0.38 y_1, \end{aligned} \quad (12)$$

as required.

(v) Similarly with (iv), we deduce that the advance is at most

$$\frac{|Z|}{\sin \beta} = \sqrt{1 + \frac{1}{\tan^2 \beta}} |Z| \leq 1.1 |Z|, \quad (13)$$

as required.

The proof of Lemma 10 is now complete.  $\square$

To finish the proof of Theorem 1, we next bound from above the total advance of each anchor point. Note that

$$\begin{aligned} x_4 &= |X| - |X \cap U_{\text{low}}| - |X \cap U_{\text{high}}| \\ &\leq \left( 1 + \frac{3}{2} \sin \phi \right) - 0.45 - 0.45 \leq 0.1 + \frac{3}{2} \sin \phi, \end{aligned}$$

and similarly

$$\begin{aligned} y_1 &= |Y| - |Y \cap U_{\text{left}}| - |Y \cap U_{\text{right}}| \\ &\leq \left( 1 + \frac{3}{2} \sin \phi \right) - 0.45 - 0.45 \leq 0.1 + \frac{3}{2} \sin \phi. \end{aligned}$$

By Lemma 10 (ii), the total advance of an anchor point due to rotations of  $\ell$  is at most 0.76.

Therefore, taking into account the inequalities in Lemma 10, the total advance of an anchor point is at most

$$\begin{aligned} &0.76 + 1.01 x_4 + 0.38 y_1 + 1.1 |Z| \\ &\leq 0.76 + 0.1013 + 0.0381 + 2.2 \sin \phi \leq 0.8997 < 0.9, \end{aligned}$$

i.e., strictly smaller than the horizontal distance of  $1 - 2w_1 = 0.9$  between the right side of  $U_{\text{left}}$  and the left side of  $U_{\text{right}}$ .

Consequently, the execution of procedure ADVANCE terminates with success and this concludes the proof of Theorem 1.  $\square$

### Remark.

The reader may wonder where the assumption  $\Gamma \subset U_3$  was needed. If  $\Gamma$  is not confined to  $U_3$ , since  $Q_2 \subset U_3$ , we know by Lemma 6 that the total length of its segments located in the exterior of  $U_3$  is small. However, these segments can pose difficulty in the analysis of ADVANCE because they could be swept by the sweep-line multiple times, backward (in the “wrong” direction) during rotation, and then forward (in

the ‘‘correct’’ direction) during translation, and then again backward and forward, etc.

#### 4. A SHARPER BOUND FOR INTER. BARRIERS BY LINEAR PROGRAMMING

In this section we prove Theorem 2, namely that the length of any interior finite segment barrier for the unit square is at least  $2 + 10^{-5}$ . Let  $w$  be a small number to be determined,  $0 < w < 1/2$ . Put  $\psi = \arctan 2w$ . Let  $\phi$  be a small angle to be determined,  $0 < \phi < \psi$ . (We will set  $w = 0.1793$  and  $\phi = 1.5589^\circ$ .) We say that a segment  $s$  is *near horizontal* (resp. *near vertical*) if the angle between the segment and the  $x$ -axis (resp.  $y$ -axis) is at most  $\phi$ . Refer to Figure 8.

Divide the unit square  $U = [0, 1]^2$  into 13 convex sub-regions (one octagon, eight triangles and four quadrilaterals) by 8 segments, each cutting off a right triangle with two shorter sides of lengths  $w$  and  $1/2$ . The height of each triangle to its hypotenuse is  $h = \frac{w/2}{\sqrt{w^2+1/4}} = \frac{1}{\sqrt{4+w^{-2}}}$ . This partition of  $U$  is suggested by our earlier Lemma 9.

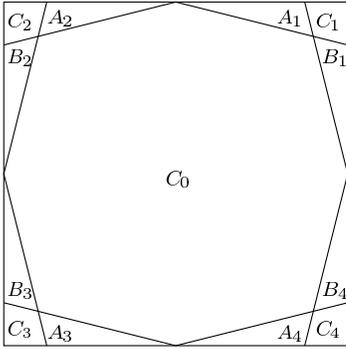


Figure 8: Partition the unit square  $U$  into 13 parts.

Let  $\Gamma$  be an interior finite segment barrier for  $U$ . Let  $X$  (resp.  $Y$ ) be the subset of  $\Gamma$  consisting of near horizontal (resp. near vertical) segments. Let  $Z = \Gamma \setminus (X \cup Y)$ . Partition each of  $X$ ,  $Y$ , and  $Z$  further into 13 subsets consisting of segments within the 13 sub-regions, respectively. We thereby obtain a partition of  $\Gamma$  into 39 subsets. In the following, we construct a linear program with 39 variables, one variable for the total length of segments in each subset, and with the goal of minimizing the sum of the 39 variables.

##### 4.1 Linear constraints based on opaque conditions of projections

For each segment  $s$  in  $\Gamma$ , denote by  $\alpha_s$  the smallest angle of rotation that brings  $s$  to either horizontal or vertical, and denote by  $\beta_s$  the smallest angle between  $s$  and a diagonal of  $U$ . Then,  $0 \leq \alpha_s \leq \pi/4$ ,  $0 \leq \beta_s \leq \pi/4$ , and  $\alpha_s + \beta_s = \pi/4$ . Denote by  $|s|_x$  (resp.  $|s|_y$ ) the length of projection of  $s$  to a horizontal (resp. vertical) side of  $U$ . Let  $|s|_{xy} = |s|_x + |s|_y$ . Denote by  $|s|_{zz}$  the total length of projection of  $s$  to the two diagonals of  $U$ . Clearly,

$$|s|_{xy} = |s| \cdot (\cos \alpha_s + \sin \alpha_s) = |s| \cdot \sqrt{2} \cos \beta_s$$

$$|s|_{zz} = |s| \cdot (\cos \beta_s + \sin \beta_s) = |s| \cdot \sqrt{2} \cos \alpha_s.$$

The total length of a horizontal side and a vertical side of  $U$  is 2. The opaque conditions in the horizontal direction

and the vertical direction require that

$$\sum_{s \in X \cup Y} |s|_{xy} + \sum_{s \in Z} |s|_{xy} \geq 2. \quad (14)$$

The total length of the two diagonals of  $U$  is  $2\sqrt{2}$ . The opaque conditions in the two directions perpendicular to the two diagonals require that

$$\sum_{s \in X \cup Y} |s|_{zz} + \sum_{s \in Z} |s|_{zz} \geq 2\sqrt{2}. \quad (15)$$

For each segment  $s$  in  $X$  or  $Y$ , we have  $\alpha_s \in [0, \phi]$  and  $\beta_s \in [\frac{\pi}{4} - \phi, \frac{\pi}{4}]$ . Thus

$$|s| \leq |s|_{xy} \leq |s| \cdot \sqrt{2} \cos \left( \frac{\pi}{4} - \phi \right),$$

$$|s| \cdot \sqrt{2} \cos \phi \leq |s|_{zz} \leq |s| \cdot \sqrt{2}.$$

For each segment  $s$  in  $Z$ , we have  $\alpha_s \in (\phi, \frac{\pi}{4}]$  and  $\beta_s \in [0, \frac{\pi}{4} - \phi)$ . Thus

$$|s| \cdot \sqrt{2} \cos \left( \frac{\pi}{4} - \phi \right) < |s|_{xy} \leq |s| \cdot \sqrt{2},$$

$$|s| \leq |s|_{zz} < |s| \cdot \sqrt{2} \cos \phi.$$

Using the upper bounds in the inequalities above, it follows from (14) and (15) that

$$|X \cup Y| \cdot \sqrt{2} \cos \left( \frac{\pi}{4} - \phi \right) + |Z| \cdot \sqrt{2} \geq 2, \quad (16)$$

$$|X \cup Y| \cdot \sqrt{2} + |Z| \cdot \sqrt{2} \cos \phi \geq 2\sqrt{2}. \quad (17)$$

##### More projections to the sides.

For each subset  $S$  of  $U$ , let  $\bar{S}$  denote  $U \setminus S$ .

For  $S = C_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$ ,

$$|X \cap S| + |Y \cap S| \cdot \sin \phi + |Z \cap S| \cdot \cos \phi \geq 1 - 2w. \quad (18)$$

For  $S = C_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4$ ,

$$|Y \cap S| + |X \cap S| \cdot \sin \phi + |Z \cap S| \cdot \cos \phi \geq 1 - 2w. \quad (19)$$

For  $S = C_0 \cup A_1 \cup A_4$  and  $S = C_0 \cup A_2 \cup A_3$ ,

$$|X \cap S| + |Y \cap S| \cdot \sin \phi + |Z \cap S| \cdot \cos \phi \geq \frac{1}{2} - w. \quad (20)$$

For  $S = C_0 \cup B_1 \cup B_2$  and  $C_0 \cup B_3 \cup B_4$ ,

$$|Y \cap S| + |X \cap S| \cdot \sin \phi + |Z \cap S| \cdot \cos \phi \geq \frac{1}{2} - w. \quad (21)$$

For  $S = \overline{B_1 \cup C_1 \cup B_4 \cup C_4}$  and  $S = \overline{B_2 \cup C_2 \cup B_3 \cup C_3}$ ,

$$|X \cap S| + |Y \cap S| \cdot \sin \phi + |Z \cap S| \cdot \cos \phi \geq 1 - w. \quad (22)$$

For  $S = \overline{A_1 \cup C_1 \cup A_2 \cup C_2}$  and  $S = \overline{A_3 \cup C_3 \cup A_4 \cup C_4}$ ,

$$|Y \cap S| + |X \cap S| \cdot \sin \phi + |Z \cap S| \cdot \cos \phi \geq 1 - w. \quad (23)$$

##### More projections to the diagonals.

For  $S = C_0 \cup A_i \cup B_i \cup C_i$ ,  $1 \leq i \leq 4$ ,

$$|(X \cup Y) \cap S| \cdot \cos \left( \frac{\pi}{4} - \phi \right) + |Z \cap S| \geq \frac{1}{4} \sqrt{2}. \quad (24)$$

For  $S = \overline{A_i \cup B_i \cup C_i}$ ,  $1 \leq i \leq 4$ ,

$$|(X \cup Y) \cap S| \cdot \cos \left( \frac{\pi}{4} - \phi \right) + |Z \cap S| \geq \frac{3}{4} \sqrt{2}. \quad (25)$$

### Projections along the hypotenuses.

For  $S = B_i \cup C_i$ ,  $1 \leq i \leq 4$ ,

$$|X \cap S| \cdot \cos(\psi - \phi) + |Y \cap S| \cdot \sin(\psi + \phi) + |Z \cap S| \geq h. \quad (26)$$

For  $S = A_i \cup C_i$ ,  $1 \leq i \leq 4$ ,

$$|Y \cap S| \cdot \cos(\psi - \phi) + |X \cap S| \cdot \sin(\psi + \phi) + |Z \cap S| \geq h. \quad (27)$$

## 4.2 Linear constraints based on the ADVANCE procedure

Now consider the ADVANCE procedure with  $a_{\text{high}}$  and  $a_{\text{low}}$  on the upper and lower sides of  $U$ , respectively, with  $x$  coordinates between  $w$  and  $1 - w$ . Put  $\beta = \arctan \frac{1}{1-2w}$ . Recall Lemma 10. By a similar analysis as in items (i) and (ii) of Lemma 10, each segment in  $X \cap (A_1 \cup A_2)$  causes a rotation that moves  $a_{\text{high}}$  for a distance at most its length times the factor  $\frac{\sin(\beta + \phi)}{\sin \beta} \cdot \frac{1}{1-w}$ , and each segment in  $X \cap (A_3 \cup A_4)$  causes an analogous movement of  $a_{\text{low}}$ . Also, as in (iii), (iv), and (v) of Lemma 10, each segment in  $X \cap C_0$ , each segment in  $Y \cap (C_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4)$ , and each segment in  $Z \cap (C_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4)$ , respectively, causes a translation that moves both  $a_{\text{high}}$  and  $a_{\text{low}}$  for a distance at most its length times  $\frac{\sin(\beta + \phi)}{\sin \beta}$ ,  $\frac{\cos(\beta - \phi)}{\sin \beta}$ , and  $\frac{1}{\sin \beta}$ .

If the two maximum movements

$$\begin{aligned} & |X \cap (A_1 \cup A_2)| \cdot \frac{\sin(\beta + \phi)}{\sin \beta} \cdot \frac{1}{1-w} \\ & + |X \cap C_0| \cdot \frac{\sin(\beta + \phi)}{\sin \beta} \\ & + |Y \cap (C_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4)| \cdot \frac{\cos(\beta - \phi)}{\sin \beta} \\ & + |Z \cap (C_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4)| \cdot \frac{1}{\sin \beta} \end{aligned}$$

and

$$\begin{aligned} & |X \cap (A_3 \cup A_4)| \cdot \frac{\sin(\beta + \phi)}{\sin \beta} \cdot \frac{1}{1-w} \\ & + |X \cap C_0| \cdot \frac{\sin(\beta + \phi)}{\sin \beta} \\ & + |Y \cap (C_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4)| \cdot \frac{\cos(\beta - \phi)}{\sin \beta} \\ & + |Z \cap (C_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4)| \cdot \frac{1}{\sin \beta} \end{aligned}$$

were both less than  $1 - 2w$ , then the ADVANCE procedure would find a line that is not blocked. Without loss of generality, assume that

$$|X \cap (A_1 \cup A_2)| \geq |X \cap (A_3 \cup A_4)|. \quad (28)$$

Then we must have

$$\begin{aligned} & |X \cap (A_1 \cup A_2)| \cdot \frac{\sin(\beta + \phi)}{\sin \beta} \cdot \frac{1}{1-w} \\ & + |X \cap C_0| \cdot \frac{\sin(\beta + \phi)}{\sin \beta} \\ & + |Y \cap (C_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4)| \cdot \frac{\cos(\beta - \phi)}{\sin \beta} \\ & + |Z \cap (C_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4)| \cdot \frac{1}{\sin \beta} \\ & \geq 1 - 2w. \end{aligned} \quad (29)$$

Similarly, assume without loss of generality that

$$|Y \cap (B_1 \cup B_4)| \geq |Y \cap (B_2 \cup B_3)|. \quad (30)$$

Then we must have

$$\begin{aligned} & |Y \cap (B_1 \cup B_4)| \cdot \frac{\sin(\beta + \phi)}{\sin \beta} \cdot \frac{1}{1-w} \\ & + |Y \cap C_0| \cdot \frac{\sin(\beta + \phi)}{\sin \beta} \\ & + |X \cap (C_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4)| \cdot \frac{\cos(\beta - \phi)}{\sin \beta} \\ & + |Z \cap (C_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4)| \cdot \frac{1}{\sin \beta} \\ & \geq 1 - 2w. \end{aligned} \quad (31)$$

## 4.3 The linear program

We construct a linear program with 32 linear constraints corresponding to inequalities (16) through (31), and 39 additional non-negativity constraints for the 39 variables. For the LP solver, we wrote a C program that uses `glp_exact` of the GNU Linear Programming Kit (GLPK 4.52) compiled with the GNU Multiple Precision Arithmetic Library (GMP 5.1.2). With parameters  $w = 0.1793$  and  $\phi = 1.5589^\circ$ , the resulting linear program yields a lower bound of  $2.0000113 \dots > 2 + 10^{-5}$ .

## 5. CONCLUSION

We have seen that while it is fairly routine to show a lower bound of 2 for the length of an arbitrary barrier for the unit square, going beyond this bound poses significant difficulties. Here we proved that any finite segment barrier for the unit square that lies in a concentric homothetic square of side length 2 has length at least  $2 + 10^{-12}$ . In particular, this bound holds for the length of any interior finite segment barrier for the unit square.

We conclude with some interesting conjectures and questions on opaque barriers that are left open.

**CONJECTURE 1.** *The square admits an optimal barrier that is interior and consists of a finite number of segments.*

If Conjecture 1 were confirmed, Theorem 1 would give a nontrivial lower bound on the length of an arbitrary barrier for the unit square. At the moment we have such a nontrivial lower bound only under a suitable locality condition, in particular for interior barriers.

- (1) Is it possible to adapt the procedure ADVANCE, or its analysis, in order to deduce a similar lower bound for arbitrary (unrestricted) barriers for the unit square?

We believe that the leftmost barrier in Figure 1 is an optimal exterior barrier for the square.

**CONJECTURE 2.** *Any optimal exterior barrier for the unit square has length 3.*

It might be interesting to note that the current best barrier for the disk is exterior, see [16, 17]. This suggests three other questions:

- (2) Can one give a characterization of the class of convex polygons whose optimal barriers are (i) interior? (ii) exterior? (iii) neither interior nor exterior?

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