Computational Geometry Column 53

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Abstract

This column is devoted to partitions of point-sets into convex subsets with interior-disjoint convex hulls. We review some partitioning problems and corresponding algorithms. At the end we list some open problems. For simplicity, in most cases we remain at the lowest possible interesting level, that is, in the plane.

Keywords: Convex partition, Steiner convex partition, Horton set, approximation algorithm, disjointness.

1 Convex partitions of point sets

Let S be a set of $n \geq 3$ points in $\mathbb{R}^2$. A convex polygon is *empty* if its interior is disjoint from S. A *convex partition* of S is a partition of the convex hull conv(S) into interior-disjoint empty convex polygons (called *tiles*) such that the vertices of the tiles are in S; see Fig. 1(left). In a *Steiner convex partition* of S the vertices of the tiles are arbitrary: they can be points in S or other *Steiner points*; see Fig. 1(middle) and Fig. 2. For example, any triangulation of S is a convex partitions of S, where the convex bodies are triangles, and so conv(S) can always be partitioned into less than $3n$ empty convex tiles.

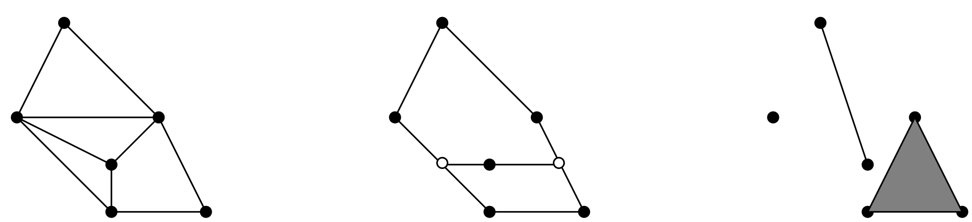

Figure 1: A convex partition with 4 parts (left), a Steiner convex partition with 2 parts (middle), and a disjoint convex partition with 3 parts (right) for the same 6-point set. Steiner points are drawn as hollow circles.

A *minimum convex partition* of S is a convex partition of S with a minimum number of tiles. Knauer and Spillner [16] showed that any n-element point set admits a convex partition with at most $\frac{15}{11}n - O(1)$ parts. On the other hand, García López and Nicolás [11] exhibited n-element point sets, $n \geq 4$, that require at least $\frac{12}{11}n - O(1)$ parts. Knauer and Spillner [16] have also obtained a $\frac{30}{11}$-factor approximation algorithm for computing a minimum convex partition for a given set $S \subset \mathbb{R}^2$, no three of which are collinear. There are also a few exact algorithms, including three fixed-parameter algorithms [9, 13, 19].

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Similarly a *minimum Steiner convex partition* of S is a Steiner convex partition with a minimum number of tiles. Given S, a simple erasing-subdivision algorithm applied to the points interior to $\mathrm{conv}(S)$ returns a Steiner convex partition with at most $\lceil (n-1)/2 \rceil$ parts [5]. The algorithm computes $\mathrm{conv}(S)$ and $\mathrm{conv}(I)$, where $I \subset S$ consists of the points interior to $\mathrm{conv}(S)$. It then repeatedly extends an arbitrary edge of $\mathrm{conv}(I)$, and then continues on the remaining set of interior points, while updating the enclosing convex region. An illustration[1] of the algorithm on a small example appears in Fig. 2(right). One can show that the above algorithm achieves a constant approximation ratio of 3. Moreover, this ratio is tight for this algorithm; see [5] for details. Since the number of parts in this partition is at most $\lceil (n-1)/2 \rceil$, obviously, the largest tile returned by the algorithm has area at least $2/n$.

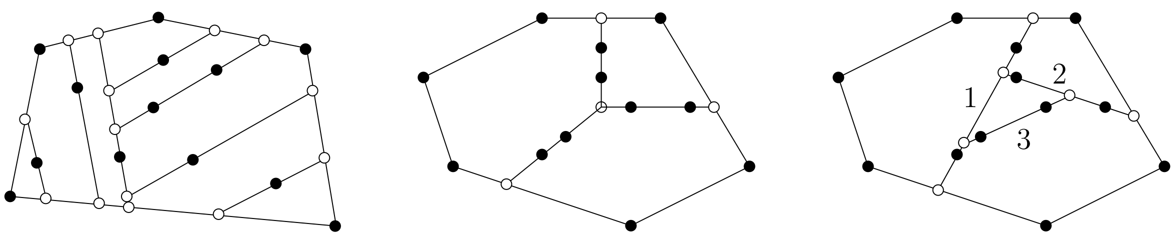

Figure 2: Steiner convex partitions with Steiner points drawn as hollow circles. Left: a convex Steiner partition of a set of 13 points. Middle: A Steiner partition of a set of 12 points into three tiles. Right: A Steiner partition of the same set of 12 points into 4 tiles, generated by the erasing-subdivision algorithm (the labels reflect the order of execution).

Many years ago, Danzer and Rogers (see [1, 2, 3, 10, 18]) asked: what is the maximum area of an empty convex body that can be found amidst any n points in the unit square? Equivalently (up to constant factors): what is the largest area of an empty triangle (or tilted rectangle) that can be found amidst any n points in the unit square? Observe that the vertices of a maximizing triangle need not be among the given points. The current best bounds are $\Omega(1/n)$ and $O((\log n)/n)$, respectively. The lower bound is trivial; interestingly enough, it can be obtained in different ways: for instance by using the erasing-subdivision algorithm described above, or by using a disk packing argument. This may be viewed as an indication that the largest area is $\omega(1/n)$. The upper bound $O((\log n)/n)$ is tight for n uniformly distributed random points in the unit square.

The erasing-subdivision algorithm for computing Steiner convex partitions can be extended to points in $\mathbb{R}^d$. Similarly to the planar case, the algorithm computes a partition with at most $\lceil (n-1)/d \rceil$ parts [5].

We say that a set of points in $\mathbb{R}^d$ is in *general position* if every k-dimensional affine subspace contains at most $k+1$ points for $0 \le k < d$. A *Horton set* [14] is a set S of n points in general position in the plane such that the convex hull of any 7 points is nonempty. Valtr [21] generalized Horton sets to d-space. For every $d \in \mathbb{N}$, there exists a minimal integer $h(d)$ with the property that for every $n \in \mathbb{N}$ there is a set S of n points in general position in $\mathbb{R}^d$ such that the convex hull of any $h(d)+1$ points in S is nonempty. Gerken [12] and Nicolas [17] showed that Horton's result [14] is the best possible, namely that $h(2) = 6$. Valtr [21] proved that $h(3) \le 22$, and in general that $h(d) \le 2^{d-1}(N(d-1)+1)$, where $N(k)$ is the product of the first k primes. Using generalized Horton sets, one can exhibit n-element point sets in $\mathbb{R}^d$ that require at least $\lceil 2(n-d-1)/h(d) \rceil$ parts in any Steiner convex partition; see [5] for details.

[1] Reproduced from [5].

2 Convex partitions of point sets into disjoint parts

Here are some partitioning problems of a slightly different kind, where we partition S rather than $\text{conv}(S)$ and we require that the convex hulls of the parts are disjoint. Let S be a set of $n \geq 3$ points in $\mathbb{R}^2$, with no three points collinear. A *disjoint convex partition* of S is a partition of S into subsets, each of which is the vertex set of a convex polygon and the convex hulls of these polygons are *geometrically* disjoint. A small example appears in Fig. 1(right).

A *minimum disjoint convex partition* of S is a disjoint convex partition of S with a minimum number of parts; see also [4, p. 366]. Observe that each part is an empty convex polygon. Urabe [20] asked what is the smallest number of parts in a minimum disjoint convex partition, over all n-element point sets.

By partitioning S into geometrically disjoint triangles (triples with disjoint convex hulls), e.g., by a sweep-line, one gets a disjoint convex partition with $\lceil n/3 \rceil$ parts. As mentioned in Section 1, for any n there exist n-element point sets with no empty heptagons [14], hence such sets require at least $n/6$ parts in any disjoint convex partition. The current best lower and upper bounds, $\lceil (n+1)/4 \rceil$ and $\lceil 5n/18 \rceil$, are due to Xu and Ding [22] and to Hosono and Urabe [15], respectively.

The following inspiring question had been raised some time ago by Aharoni and Saks; see [8]. Its connection with disjoint convex partitions will become apparent shortly. Given any two-colored set of n points in the plane, with no three points collinear, is it always possible to match all but a constant number of points, by straight-line non-crossing edges connecting points of the same color? The parity of n or of the color classes is irrelevant here, as one can make all of them even by ignoring $O(1)$ of the n points. It is easy to see that even with four points, two of each color, sometimes two points need to remain unmatched, as in Fig. 3(left). An examination of the surprising 18-point example found by Eli Berger (communicated by M. Saks; see also [8]) illustrated in Fig. 3(middle) shows that at least 4 points remain unmatched in that configuration.

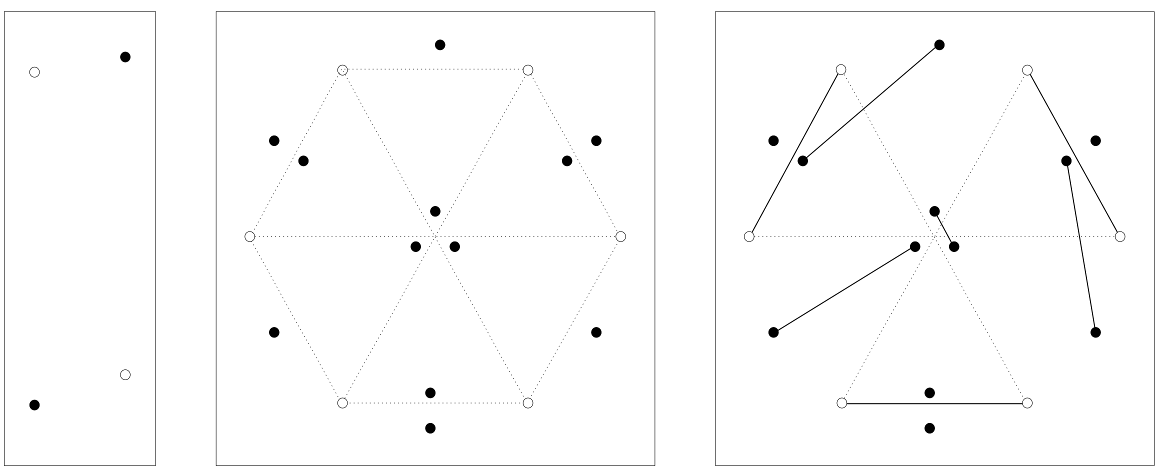

Figure 3: A 4-point example (left) and a 18-point example (middle); all possible collinearities in the 18-point example are resolved by small perturbations. Right: a (maximum) matching of 14 out of the 18 points.

As it turns out [8], this trend continues and the answer to this question is in the negative: there exist arbitrary large two-colored point sets such that any matching with the required properties matches fewer than $(1-\delta)n$ points, for some positive constant δ. The upper bound was obtained by a probabilistic construction. The current best bounds on δ were established in [6], where it was shown that one can always match $6n/7 - O(1)$ points, and sometimes not more than $94n/95 + O(1)$ points. Observe that any matching as above leads to a disjoint convex partition of the point-set into monochromatic parts (pairs or singletons) with pairwise-disjoint convex hulls.

The matching problem with monochromatic edges has lead to the following development, where the monochromatic parts are not restricted in size. One can show [7] that any two-colored set of n points in the plane, with no three points collinear, can be partitioned into $\lceil \frac{n+1}{2} \rceil$ monochromatic parts, whose convex hulls are disjoint. This is in contrast to the partitions obtained from matchings, where sometimes $n/2 + \Omega(n)$ parts are necessary. Moreover, the $\lceil \frac{n+1}{2} \rceil$ bound cannot be improved. Similarly, one may try to partition a two-colored set of n points in $\mathbb{R}^3$ into a minimum number of monochromatic parts with disjoint convex hulls. It is known that $n/3 + O(1)$ parts always suffice, but the current best lower bound is only sublinear [7].

3 Open problems

Interesting questions remain open regarding the structure of optimal (Steiner or non-Steiner) convex partitions and the computational complexity of computing such partitions. Other questions relate to the problem of finding the largest empty convex body in the presence of points. We also include other problems regarding convex partitions of point-sets into (geometrically) disjoint parts.

(1) Is there a polynomial-time algorithm for computing a minimum convex partition of a given set of n points in the plane? Can the ratio 30/11 approximation for points in general position be improved?

(2) Is there a polynomial-time algorithm for computing a minimum Steiner convex partition of a given set of n points in the plane? Can the ratio 3 approximation for points in general position be improved?

(3) Is there a constant-factor approximation algorithm for the minimum Steiner convex partition of an arbitrary point set in the plane (without the general position restriction)?

(4) What is the answer to the question of Danzer and Rogers? Is the largest area of an empty triangle that can be found amidst any n points in the unit square $\omega(1/n)$?

(5) Find a tighter estimate for the minimum number of parts in a Steiner convex partition of n points in general position in $\mathbb{R}^d$. Observe that the gap between the current bounds grows exponentially in the dimension d.

(6) What is the answer to the partitioning problem of Urabe? Can an optimal disjoint convex partition be efficiently computed?

(7) What is the quantitative answer to the matching problem of Aharoni and Saks for two-colored point-sets? Can one always match 95% of the points (i.e., $19n/20 - o(n)$ points) using monochromatic non-crossing edges? Can a maximum-cardinality matching be efficiently computed?

(8) What is the minimum number of monochromatic parts whose convex hulls are pairwise disjoint, into which any 2-colored set of n points in $\mathbb{R}^3$ (no 4 coplanar) can be partitioned? Is it $o(n)$? Can an optimal partition be efficiently computed (in the plane or in 3-space)?

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References

[1] N. Alon, I. Bárány, Z. Füredi, and D. Kleitman, Point selections and weak ε-nets for convex hulls, *Combinatorics, Probability & Computing*, **1** (1992), 189–200.

[2] R. P. Bambah and A. C. Woods, On a problem of Danzer, *Pacific Journal of Mathematics*, **37(2)** (1971), 295–301.

[3] J. Beck and W. Chen, *Irregularities of Distributions*, vol. 89 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1987.

[4] P. Braß, W. Moser, and J. Pach, *Research Problems in Discrete Geometry*, Springer, New York, 2005.

[5] A. Dumitrescu, S. Har-Peled, and Cs. D. Tóth, Minimum convex partitions and maximum empty polytopes, *Proceedings of the 13th Scandinavian Symposium and Workshops on Algorithm Theory* (SWAT 2012), Helsinki, Finland, July 2012, to appear in LNCS. `arXiv:1112.1124`.

[6] A. Dumitrescu and R. Kaye, Matching colored points in the plane: some new results, *Computational Geometry: Theory and Applications*, **19(1)** (2001), 69–85.

[7] A. Dumitrescu and J. Pach, Partitioning colored point sets into monochromatic parts, *International Journal of Computational Geometry & Applications*, **12(5)** (2002), 401–412.

[8] A. Dumitrescu and W. Steiger, On a matching problem in the plane, *Discrete Mathematics*, **211(1-3)** (2000), 183–195.

[9] T. Fevens, H. Meijer, and D. Rappaport, Minimum convex partition of a constrained point set, *Discrete Applied Mathematics*, **109(1-2)** (2001), 95–107.

[10] Z. Füredi and J. Pach, Traces of finite sets: extremal problems and geometric applications, in *Extremal Problems for Finite Sets (P. Frankl, Z. Füredi, G. Katona, D. Miklós, editors)*, vol. 3 of Bolyai Society Mathematical Studies, Budapest, 1994, pp. 251–282.

[11] J. García López and C. Nicolás, Planar point sets with large minimum convex partitions, *Proceedings of the 22nd European Workshop on Computational Geometry*, Delphi, Greece, 2006, pp. 51–54.

[12] T. Gerken, Empty convex hexagons in planar point sets, *Discrete & Computational Geometry*, **39(1-3)** (2008), 239–272.

[13] M. Grantson and C. Levcopoulos, A fixed-parameter algorithm for the minimum number convex partition problem, *Proceedings of the Japanese Conference on Discrete and Computational Geometry*, vol. 3742 of LNCS, Springer, 2005, pp. 83–94.

[14] J. Horton, Sets with no empty convex 7-gons, *Canadian Mathematical Bulletin*, **26** (1983), 482–484.

[15] K. Hosono and M. Urabe, On the number of disjoint convex quadrilaterals for a planar point set, *Computational Geometry: Theory and Applications*, **20** (2001), 97–104.

[16] C. Knauer and A. Spillner, Approximation algorithms for the minimum convex partition problem, *Proceedings of the 10th Scandinavian Workshop on Algorithm Theory* (SWAT 2006), vol. 4059 of LNCS, Springer, 2006, pp. 232–241.

[17] C. M. Nicolas, The empty hexagon theorem, *Discrete & Computational Geometry*, **38(2)** (2007), 389–397.

[18] J. Pach and G. Tardos, Piercing quasi-rectangles: on a problem of Danzer and Rogers, *Proceedings of the Algorithms and Data Structures Symposium* (WADS 2011), vol. 6844 of LNCS, Springer, 2011, p. 654.

[19] A. Spillner, A fixed parameter algorithm for optimal convex partitions, *Journal of Discrete Algorithms*, **6(4)** (2008), 561–569.

[20] M. Urabe, On a partition in convex polygons, *Discrete Applied Mathematics*, **64** (1996), 179–191.

[21] P. Valtr, Sets in $\mathbb{R}^d$ with no large empty convex subsets, *Discrete Mathematics*, **108(1-3)** (1992), 115–124.

[22] C. Xu and R. Ding, On the empty convex partition of a finite set in the plane, *Chinese Annals of Mathematics*, **23B** (2002), 487–494.