Watchman Routes for Lines and Line Segments

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Abstract

Given a set $\mathcal{L}$ of non-parallel lines in the plane, a watchman route (tour) for $\mathcal{L}$ is a closed curve contained in the union of the lines in $\mathcal{L}$ such that every line is visited (intersected) by the route; we similarly define a watchman route (tour) for a connected set $\mathcal{S}$ of line segments. The watchman route problem for a given set of lines or line segments is to find a shortest watchman route for the input set, and these problems are natural special cases of the watchman route problem in a polygon with holes (a polygonal domain).

In this paper, we show that the problem of computing a shortest watchman route for a set of n non-parallel lines in the plane is polynomially tractable, while it becomes NP-hard in 3D. We give an alternative NP-hardness proof of this problem for line segments in the plane and obtain a polynomial-time approximation algorithm with ratio $O(\log^3 n)$. Additionally, we consider some special cases of the watchman route problem on line segments, for which we provide exact algorithms or improved approximations.

Keywords: Watchman route, dynamic programming, NP-hardness.

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1 Introduction

In 1973, Victor Klee asked what is the minimum number of stationary guards that can watch over all the paintings that hang in a gallery with n walls. The answer was given by Chvátal [9], who proved that $\lfloor \frac{n}{3} \rfloor$ guards are always sufficient and sometimes necessary to cover a polygon with n vertices. Over the last few decades, numerous variations of the above *art gallery problem* have been studied, including mobile guards, guards with limited visibility or mobility, guarding for special classes of polygons, etc.; see O'Rourke's monograph [30], the survey articles [27, 34, 42], and recent papers [1, 2, 13, 39].

The watchman route problem has been introduced by Chin and Ntafos [7, 8]. A *watchman route* in a polygon P (either a simple polygon or a polygon with holes, also known as a polygonal domain) is a closed curve inside P such that every point in P is visible from at least one point of the route. For simple polygons, the shortest route can be found in $O(n^4 \log n)$ time [6, 10, 36], and a 2-approximation can be computed in linear time [37]. For polygons with holes the problem cannot be approximated in polynomial time to within a factor of $c \log n$, for a suitable constant $c > 0$, assuming that $\mathrm{P} \neq \mathrm{NP}$ [28]. For the special case of orthogonal polygons and orthogonal visibility an $O(\log n)$-approximation algorithm has been reported in [26], and, for the general case, the first polynomial-time approximation algorithm (with factor $O(\log^2 n)$) has been proposed by Mitchell [28].

In this paper, we study a natural special case of the watchman route problem in polygons with holes. We consider watchman routes for a collection of lines or for a collection of line segments. One can view the lines or the line segments as streets in a city. A watchman route is constrained to lie on the road network, i.e., the union of the lines or line segments. A line (or a line segment) can be "seen" in both directions from any point incident to it, in particular from any vertex of the arrangement of lines (or line segments). Consequently, a watchman route for a collection of lines (or line segments) is a polygonal route. See Fig. 1 for an illustration.

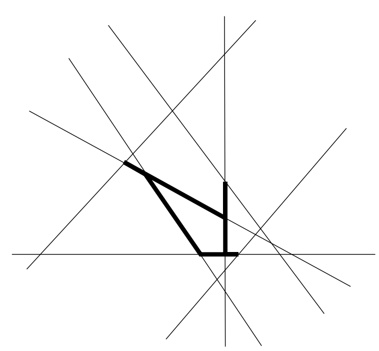

Fig. 1: A watchman route (in bold) for a set of seven lines (or line segments); three subsegments (spikes) are traversed by the route in both directions.

We say that the arrangement $\mathcal{A}(\mathcal{L})$ of a set of lines $\mathcal{L}$ is *connected*, or for short, $\mathcal{L}$ is connected, if there exists a path $\xi \subset \cup_{l \in \mathcal{L}} l$ from any point $p \in l$ to any other point $p' \in l'$, for any $l, l' \in \mathcal{L}$. Similarly, we define a connected arrangement of a set of segments. Observe that a set of lines $\mathcal{L}$ is connected if and only if not all lines are parallel (i.e., there exist two non-parallel lines in $\mathcal{L}$). Formally, the watchman route problem for lines or line segments is defined as follows.

The watchman route problem for lines (WRL): Given a connected set $\mathcal{L}$ of lines in $\mathbb{R}^d$, find a shortest watchman route for $\mathcal{L}$.

The watchman route problem for segments (WRS): Given a connected set $\mathcal{S}$ of line segments in $\mathbb{R}^d$, find a shortest watchman route for $\mathcal{S}$.

Here, we focus primarily on the problems WRL and WRS in the plane $\mathbb{R}^2$ ($d = 2$). Previously WRS appears to have been only considered for arrangements of axis-aligned segments (called grids) [43, 44]; this variant has been introduced by Hoffmann [23]. Xu and Brass [43, 44] proved the NP-hardness of WRS by a reduction from the connected vertex cover problem in planar graphs with maximum degree four. Other variants of the art gallery problem for line segments have been studied in [4, 18, 25, 29, 30, 40, 41], to mention just a few.

Our results. In Section 2, we provide a polynomial-time algorithm for computing a shortest watchman route for a set of n non-parallel lines in the plane, and show that the watchman route problem for orthogonal lines in 3D is NP-hard. In Section 3, we show that the watchman route problem for axis-aligned line segments in the plane is also NP-hard (with a simpler proof than [43, 44]), and give an approximation algorithm with ratio $O(\log^3 n)$ for connected sets of segments in any dimension. Additionally, we show that an approximation algorithm with a constant ratio exists for certain special cases of the watchman route problem for segments, and we show how to compute an optimal watchman route for "outerplanar grids" (defined in Subsection 3.4).

A connected set of lines or segments can be thought of as a polygon with holes, consisting of very thin corridors. So in particular, our result for lines provides a polynomial time optimal algorithm for the watchman route problem in a restricted subclass of polygons with holes, while the result for line segments provides a polynomial time $O(\log^3 n)$-approximation algorithm for another wider subclass of polygons with holes; recall that the approximation algorithm in [26] applies only to orthogonal polygons with holes, under orthogonal visibility. In addition, our result for lines shows that some instances of TSP with neighborhoods (TSPN) *and with obstacles* are polynomially solvable (the obstacles are the open faces of the input line arrangement), while obviously TSPN without obstacles is generally NP-hard. It is worth mentioning that TSPN for a set of lines in the plane (with no obstacles) is solvable in polynomial time as a special case of the watchman route in a simple polygon [6, 10, 24, 36, 38].

Definitions and notations. For a set of lines $\mathcal{L}$ (resp. a set of line segments $\mathcal{S}$), let $V(\mathcal{A}(\mathcal{L}))$ (resp. $V(\mathcal{A}(\mathcal{S}))$) denote the set of vertices of the arrangement $\mathcal{A}(\mathcal{L})$ (resp. $\mathcal{A}(\mathcal{S})$) formed by the lines in $\mathcal{L}$ (resp. line segments in $\mathcal{S}$). Next, let $G(\mathcal{L})$ (resp. $G(\mathcal{S})$) denote the weighted planar graph with vertex set $V(\mathcal{A}(\mathcal{L}))$ (resp. $V(\mathcal{A}(\mathcal{S}))$) whose edges connect successive vertices on the lines in $\mathcal{L}$ (resp. the line segments in $\mathcal{S}$); the weight of an edge is the Euclidean distance between the corresponding vertices along the connecting line (resp. segment). For $s, t \in V(\mathcal{A}(\mathcal{L}))$ (resp. $s, t \in V(\mathcal{A}(\mathcal{S}))$), let $\pi(s,t) = \pi_G(s,t)$ denote a shortest path connecting s and t in $G(\mathcal{L})$ (resp. in $G(\mathcal{S})$); the length of $\pi(s,t)$ is denoted by $|\pi(s,t)|$.

For a route $\mathcal{R}$, $\mathrm{conv}(\mathcal{R})$ denotes its convex hull, and $|\mathcal{R}|$ denotes its length. For a set of lines $\mathcal{L}$ (resp. a set of line segments $\mathcal{S}$), let $OPT(\mathcal{L})$ (resp. $OPT(\mathcal{S})$) denote an optimal watchman route for $\mathcal{L}$ (resp. $\mathcal{S}$).

An arrangement of axis-parallel segments is called a *grid*.

2 The watchman route problem for lines

We first develop a polynomial-time algorithm for computing a shortest watchman route for a set $\mathcal{L}$ of n lines in the plane. Our approach is based upon a dynamic programming technique. We then show that the watchman route problem for lines in 3D becomes NP-hard, even for axis-parallel lines.

2.1 Lines in the plane

The input is a set $\mathcal{L}$ of n lines, not all parallel to each other; otherwise, the line arrangement is not connected, and no watchman route exists. We can assume without loss of generality that no line in $\mathcal{L}$ is horizontal. Our algorithm is based on the following crucial observation: Since a watchman route is connected, a line $\ell \in \mathcal{L}$ intersects a watchman route $\mathcal{R}$ if and only if it intersects the convex hull of $\mathcal{R}$. Thus, in order to compute an optimal watchman tour $OPT(\mathcal{L})$ for $\mathcal{L}$, we only need to compute $\text{conv}(OPT(\mathcal{L}))$. This is done by solving the *minimum convex hull* problem, defined as follows.

The minimum convex hull problem (MCH): Given a set $\mathcal{L}$ of n lines in the plane, not all parallel, compute a minimum-length cyclic sequence $(v_1, \ldots, v_h, v_1)$ of vertices $v_i \in V(\mathcal{A}(\mathcal{L}))$ in convex position, such that every line in $\mathcal{L}$ intersects the convex polygon $(v_1, \ldots, v_h)$, where the *length of* $(v_1, \ldots, v_h, v_1)$ is defined as $\sum_{i=1}^{h} |\pi(v_i, v_{i+1})|$, with $v_{h+1} = v_1$.

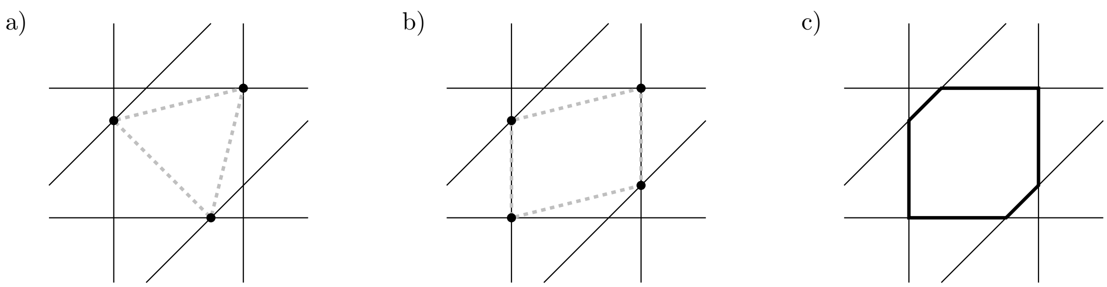

Fig. 2: There can be multiple solutions to MCH, e.g., one with $h = 3$ (left) and one with $h = 4$ (middle), both yielding the same optimal watchman tour (right). Observe that both the left and middle cyclic sequence optimizing MCH are strictly contained in the convex hull of the shortest watchman tour.

The following lemma formalizes the relationship between MCH and the watchman route problem for lines (WRL).

Lemma 1. *For a set $\mathcal{L}$ of (non-parallel) lines in the plane, a solution to MCH for $\mathcal{L}$ yields a solution to WRL for $\mathcal{L}$.*

Proof. Let $(v_1, \ldots, v_h, v_1)$ be a solution to MCH for $\mathcal{L}$. Consider the route $\mathcal{R}$ that results from the concatenation of the shortest paths $\pi(v_1, v_2), \ldots, \pi(v_{h-1}, v_h)$, and $\pi(v_h, v_1)$. We claim that $\mathcal{R}$ is a shortest watchman route for $\mathcal{L}$.

First, since each line $\ell \in \mathcal{L}$ intersects the convex polygon $(v_1, \ldots, v_h)$, each line $\ell \in \mathcal{L}$ must intersect the route $\mathcal{R}$. (Otherwise, a line ℓ would separate some vertex v_i from some other vertex v_j, without intersecting $\mathcal{R}$ — a contradiction to the connectedness of $\mathcal{R}$.) Thus, $\mathcal{R}$ is a watchman route for $\mathcal{L}$, and so the length of $(v_1, \ldots, v_h, v_1)$, equal to $|\mathcal{R}|$, is at least $|OPT(\mathcal{L})|$.

Next, consider an optimal route $OPT(\mathcal{L})$, which is a solution to WRL for $\mathcal{L}$. Since the vertices of $OPT(\mathcal{L})$ are vertices of the arrangement $\mathcal{A}(\mathcal{L})$, we know that $\text{conv}(OPT(\mathcal{L}))$ has vertices in the set $V(\mathcal{A}(\mathcal{L}))$. Since each $\ell \in \mathcal{L}$ intersects the route $OPT(\mathcal{L})$, we know that each $\ell \in \mathcal{L}$ also

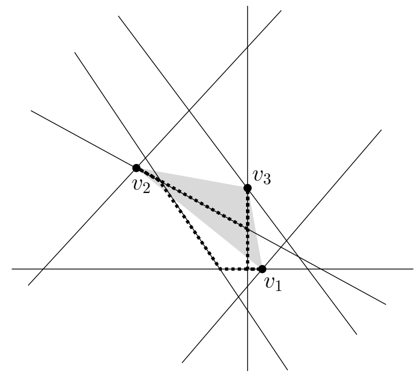

Fig. 3: Joining the consecutive vertices of $C = (v_1, v_2, v_3, v_1)$ by using shortest paths yields a non-convex (non-simple) polygon.

intersects $\text{conv}(OPT(\mathcal{L}))$. Thus, the vertices of $\text{conv}(OPT(\mathcal{L}))$ form a cyclic sequence of vertices in $V(\mathcal{A}(\mathcal{L}))$ that is feasible for MCH, and the length of this sequence is exactly $|OPT(\mathcal{L})|$, since $OPT(\mathcal{L})$ must use shortest paths to link any two consecutive vertices of $\text{conv}(OPT(\mathcal{L}))$ (otherwise, the route could be shortened while still visiting every line of $\mathcal{L}$).

Consequently, on the one hand, no solution to MCH can have the length smaller then $|OPT(\mathcal{L})|$ since otherwise, $\mathcal{R}$ is a watchman route of cost smaller then $|OPT(\mathcal{L})|$ — a contradiction with optimality of $OPT(\mathcal{L})$. On the other hand, $\text{conv}(OPT)$ is a valid candidate of length $|OPT(\mathcal{L})|$ for a solution to MCH. We conclude that $\mathcal{R}$ is a shortest watchman route for $\mathcal{L}$. $\square$

Observe that there can be several solutions to MCH, all having the same length but having different sequences of vertices in convex position, all corresponding to the same optimal tour $OPT(\mathcal{L})$. In particular, note that a cyclic sequence $C = (v_1, \ldots, v_h, v_1)$ optimizing MCH, i.e., the convex polygon $Q = (v_1, \ldots, v_h)$, can be strictly contained in the convex hull Q' of the corresponding route (obtained by using shortest paths to link the vertices of C), see Fig. 2; in this case, there are multiple solutions to MCH — Q' also induces a solution to MCH, with the same length as C. On the other hand, the route obtained by using shortest paths to link the vertices of C may not be convex or even contained in the convex polygon Q, see Fig. 3.

2.1.1 Dynamic programming for MCH

The goal is finding a minimum-length cyclic sequence $C = (v_1, \ldots, v_h, v_1)$, with vertices in $V(\mathcal{A}(\mathcal{L}))$ in convex position, such that every line $\ell \in \mathcal{L}$ intersects the convex polygon $Q = (v_1, \ldots, v_h)$. If $h = 1$, Q consists of a single vertex of $\mathcal{A}(\mathcal{L})$, and this happens if and only if all lines in $\mathcal{L}$ are incident to a single point, which constitutes a (trivial) zero-length solution to MCH. If there is no trivial solution to MCH, we find a minimum-length cyclic sequence C with $h \geq 2$.

If $h = 2$, Q consists of a line segment; this case is also trivial to check: we enumerate all pairs $\{v_1, v_2\}$ of vertices in $V(\mathcal{A}(\mathcal{L}))$, and check whether every $\ell \in \mathcal{L}$ intersects the (closed) segment v_1v_2, and determine a pair having the shortest length $|\pi(v_1, v_2)| + |\pi(v_2, v_1)| = 2|\pi(v_1, v_2)|$ among such candidate pairs.

Assume now that $h \geq 3$, no line in $\mathcal{L}$ is horizontal, and all vertices in $V(\mathcal{A}(\mathcal{L}))$ have distinct y-coordinates. The algorithm examines all possible choices of the lowest (with minimum y-coordinate) vertex of C. Let v_1 be such a lowest vertex, let l_1 be the horizontal line through v_1, and let h_1 be the closed half-plane above l_1. Since no line in $\mathcal{L}$ is horizontal, $l_1 \notin \mathcal{L}$ and all lines in $\mathcal{L}$ intersect the half-plane h_1. Assume $V(\mathcal{A}(\mathcal{L})) \cap h_1 = \{v_1, v_2, \ldots, v_{m-1}\}$, for some $m \geq 4$, where vertices

$v_2, \ldots, v_{m-1}$ are ordered according to increasing angle with respect to the horizontal ray directed to the left, from v_1 (in clockwise order around v_1). In case of ties, we order vertices by increasing distance from v_1. Next, set a new element $v_m := v_1$, and let $V^* = (V(\mathcal{A}(\mathcal{L})) \cap h_1) \cup \{v_m\}$. (We emphasize that although v_m and v_1 correspond to the same point in the plane, they are distinct elements, and so $|V^*| = m$.)

For above ordering scheme, if $C = (v_1, v_{i_2}, v_{i_3}, \ldots, v_m)$ is a solution to MCH, since no three distinct vertices of C are collinear, we have $1 < i_2 < i_3 < \cdots < m$. Consequently, only ordered pairs (v_j, v_k) of vertices, where $1 \leq j < k \leq m$ and either $j \neq 1$ or $k \neq m$, are candidate hull edges for this solution; in particular, for $1 < j < k < m$, we can (and will) restrict ourselves to pairs (v_j, v_k) for which v_j and v_k are not collinear with $v_1 = v_m$.

For $1 \leq j < k \leq m$ and either $j \neq 1$ or $k \neq m$, let $l_{j,k}$ denote the directed line through v_j and v_k (directed from v_j to v_k), let $h_{j,k}$ denote the (closed) half-plane on the right of $l_{j,k}$, and let $C_{j,k} = h_1 \cap h_{j,k}$ denote the cone that is the intersection of h_1 and $h_{j,k}$. Depending on whether the slope of $v_j v_k$ is positive or negative, if $j \neq 1$ and $k \neq m$, then the apex of the cone $C_{j,k}$ will lie to the left or to the right of v_1, respectively; Fig. 4 illustrates both cases. Next, let $R_{1,k} = \{v_1\}$ for $1 < k < m$, and let $R_{j,k}$, for $1 < j < k \leq m$, denote the (possibly unbounded) closed triangular region $C_{j,k} \cap h_{j,m}$. Finally, for $1 \leq j < k \leq m$ and either $j \neq 1$ or $k \neq m$, let $\mathcal{L}_{j,k}$ denote the subset of lines $\ell \in \mathcal{L}$ that intersect $R_{j,k}$ but do not intersect the (closed) line segment $v_j v_m$. (Notice that $\mathcal{L}_{1,k} = \emptyset$.)

Clearly, $C_{j,k}$, $R_{j,k}$ and $\mathcal{L}_{j,k}$ depend also on the choice of v_1; however, for notational convenience, we omit showing the explicit dependence. For the remainder of our algorithm description, we fix a particular choice of v_1; the outer loop of the algorithm iterates over all $O(n^2)$ choices of v_1.

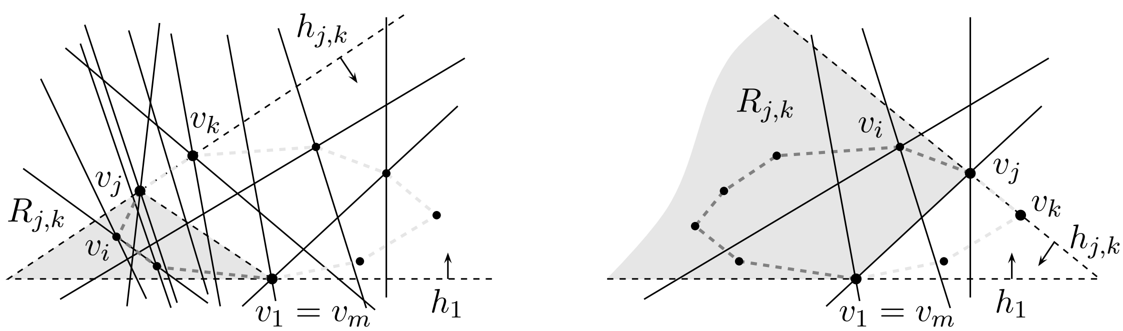

Fig. 4: SubProblem(v_1, v_j, v_k), $1 \leq j < k \leq m$, either $j \neq 1$ or $k \neq m$; $v_i \in R_{j,k} \setminus l_{j,k}$.

We say that an (ordered) pair (v_j, v_k) of vertices in V^*, for $1 \leq j < k \leq m$ and either $j \neq 1$ or $k \neq m$, is *eligible* if every line $\ell \in \mathcal{L}$ intersects the cone $C_{j,k}$. If (v_j, v_k) is not eligible, it does not need to be considered as a candidate edge of the convex polygon Q that is the desired solution to MCH: there is some line in $\mathcal{L}$ that does not intersect the cone $C_{j,k}$, and therefore, since $Q \subset C_{j,k}$, it does not intersect any convex polygon Q having the edge $v_j v_k$.

For each eligible pair (v_j, v_k), $1 \leq j < k \leq m$ and either $j \neq 1$ or $k \neq m$, SubProblem(v_1, v_j, v_k) is defined as follows.

SubProblem(v_1, v_j, v_k): Compute a minimum-length convex (right-turning) chain from v_1 to v_j such that the chain lies within the region $R_{j,k}$ and it intersects every line $\ell \in \mathcal{L}_{j,k}$. (The lines in $\mathcal{L}_{j,k}$ are the responsibility of the subproblem to visit.)

Next, for an eligible pair (v_j, v_k), let $f(j, k)$ denote the minimum length of a chain from v_1 to v_j that solves SubProblem(v_1, v_j, v_k); if such a chain does not exist or (v_j, v_k) is not eligible, then we set $f(j,k) = \infty$. Note that $f(1,k) = 0$ if (v_1, v_k) is eligible. Our overall problem is to find an eligible pair (v_j, v_m) such that $f(j,m) + |\pi(v_j, v_m)|$ is minimized over all eligible pairs (v_j, v_m); in such a case, $v_j v_m = v_j v_1$ is the last edge of the convex polygon Q formed by an optimal cyclic sequence, starting at v_1 and going in the clockwise manner around Q, returning to $v_m = v_1$.

The dynamic programming recursion (Bellman equation) is defined as follows. The base of the recursion is $f(1,k) = 0$, if (v_1, v_k) is eligible, and $f(j,k) = \infty$, if (v_j, v_k) is not eligible ($1 \le j < k \le m$ and either $j \ne 1$ or $k \ne m$). Next, for an eligible pair (v_j, v_k), $1 < j < k \le m$,

$$f(j,k) = \min_{i \in I_{j,k}} (f(i,j) + |\pi(v_i, v_j)|), \tag{1}$$

where $I_{j,k}$ is the set of all indices i with $1 \le i < j$ such that

(i) $v_i \in R_{j,k} \setminus l_{j,k}$ (which enforces convexity of the chain), and

(ii) each line $\ell \in \mathcal{L}_{j,k} \setminus \mathcal{L}_{i,j}$ intersects the (right-open) segment $v_i v_j$.

In particular, note that $f(j,k) = |\pi(v_1, v_j)|$ if $\mathcal{L}_{j,k} = \emptyset$ (and (v_1, v_j) is eligible), so that a one-edge chain from v_1 to v_j suffices to meet all lines that are the responsibility of the subproblem. If $I_{j,k} = \emptyset$, then $f(j,k) = \infty$. Also, recall that we defined $f(j,k) = \infty$, if (v_j, v_k) is not eligible. (In particular, the minimization over $i \in I_{j,k}$ will never select an i for which (v_i, v_j) is not eligible, since $f(i,j) = \infty$ for such an index i.)

We tabulate the values $f(i,j)$ in clockwise angular order around v_1, so that the values $f(i,j)$, for $1 \le i < j < k \le m$, are known by the time they are needed to compute $f(j,k)$. We pre-compute and tabulate eligibility for all pairs of vertices (for a fixed choice of v_1); for each of the $O(n^6)$ choices of v_1, v_j, v_k, eligibility of (v_j, v_k) is determined in $O(n)$ time by testing each of n lines in $\mathcal{L}$ for intersection with $C_{j,k}$. Note that condition (i) in the definition of $I_{j,k}$ can be easily checked in $O(1)$ time per candidate v_i. For efficiently testing condition (ii) in $O(1)$ time per candidate v_i, thus in $O(n^2)$ total time, some additional preprocessing is done.

Specifically, let $l_1^- \subset l_1$ (resp. $l_1^+ \subset l_1$) be the left (resp. right) half-line of l_1 with the endpoint at v_1. Let $\mathcal{L}_{j,k}^-$ and $\mathcal{L}_{j,k}^+$ denote the subsets of lines in $\mathcal{L}_{j,k}$ that intersect l_1^- and l_1^+, respectively; note that $\mathcal{L}_{j,k}^- \cup \mathcal{L}_{j,k}^+ = \mathcal{L}_{j,k}$ and $\mathcal{L}_{j,k}^- \cap \mathcal{L}_{j,k}^+ = \emptyset$, and if the slope of $l_{j,k}$ is positive, then $\mathcal{L}_{j,k}^+ = \emptyset$. Next, let $X_{j,k}^-$ denote the shortest horizontal line segment contained in l_1^- that includes all points of intersection between lines in $\mathcal{L}_{j,k}^-$ and l_1^-. And, let $\alpha_{j,k}^+$ denote the smallest angle among those that lines in $\mathcal{L}_{j,k}^+$ form with l_1^+, in the counterclockwise manner; if $\mathcal{L}_{j,k}^+ = \emptyset$, we set $\alpha_{j,k}^+ := 180°$. Clearly, all the above data can be pre-computed and tabulated in $O(n)$ time per choice of tuple (v_1, v_j, v_k), thus in $O(n^6) \cdot O(n) = O(n^7)$ time. Now, we distinguish the following three cases.

Case 1: both lines $l_{i,j}$ and $l_{j,k}$ intersect l_1^- (Fig. 5). Observe that $\mathcal{L}_{j,k} \setminus \mathcal{L}_{i,j}$ consists of lines that intersect either the right-open line segment $v_i v_j$ or the right-open line segment $p_{j,k} p_{i,j}$, where $p_{j,k} = l_{j,k} \cap l_1^-$ and $p_{i,j} = l_{i,j} \cap l_1^-$. Note also that there is a line $\ell \in \mathcal{L}_{j,k} \setminus \mathcal{L}_{i,j}$ that intersects the right-open line segment $p_{j,k} p_{i,j}$ if and only if the left endpoint of $X_{j,k}^-$ is to the left of $p_{i,j}$. Therefore, when handling a candidate v_i, all we need is to check whether the left endpoint of $X_{j,k}^-$ is to the left of $p_{i,j}$.

Case 2: line $l_{i,j}$ intersects l_1^- and line $l_{j,k}$ intersects l_1^+. (Fig. 6). Observe that $\mathcal{L}_{j,k} \setminus \mathcal{L}_{i,j}$ consists of lines that either intersect the right-open line segment $v_i v_j$, or intersect the half-line l_1^- to the left of $p_{i,j} = l_{i,j} \cap l_1^-$ or are in $\mathcal{L}_{i,j}^+$. Therefore, when handling a candidate v_i, all we need is to check whether (a) the left endpoint of $X_{j,k}^-$ is to the left of $p_{i,j}$ and (b) whether $\mathcal{L}_{i,j}^+ \ne \emptyset$.

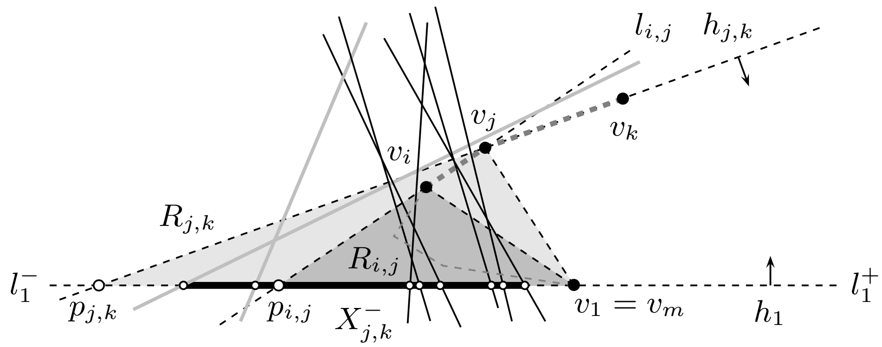

Fig. 5: Selecting the vertex v_i in the recursion to solve SubProblem(v_1, v_j, v_k): Both lines $l_{i,j}$ and $l_{j,k}$ intersect l_1^-. Since the left endpoint of $X_{j,k}^-$ is to the left of $p_{i,j}$, not all lines in $\mathcal{L}_{j,k} \setminus \mathcal{L}_{i,j}$ intersect $v_i v_j$ — the relevant lines are marked with gray — and thus, condition (ii) is not satisfied.

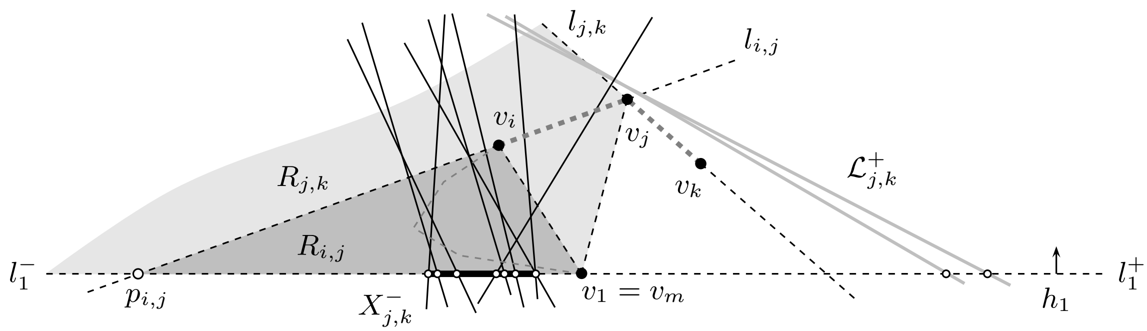

Fig. 6: Selecting the vertex v_i in the recursion to solve SubProblem(v_1, v_j, v_k): Line $l_{i,j}$ intersects l_1^- and line $l_{j,k}$ intersects l_1^+. Since $\mathcal{L}_{j,k}^+ \neq \emptyset$, not all lines in $\mathcal{L}_{j,k} \setminus \mathcal{L}_{i,j}$ intersect $v_i v_j$ — the relevant lines are marked with gray — and thus, condition (ii) is not satisfied.

Case 3: both lines $l_{i,j}$ and $l_{j,k}$ intersect l_1^+. (Fig. 7). Observe that $\mathcal{L}_{j,k} \setminus \mathcal{L}_{i,j}$ consists of lines that either intersect the right-open line segment $v_i v_j$ or are in $\mathcal{L}_{j,k}^+$ and their (counterclockwise) angle with l_1^+ is not larger than the angle that $l_{i,j}$ forms with l_1^+. In order to check whether any line in $\mathcal{L}_{i,j}^+ \neq \emptyset$ satisfies the latter subcondition all we need is check whether $\alpha_{j,k}^+$ is not larger than the angle that $l_{i,j}$ forms with l_1^+.

Consequently, condition (ii) can be verified in $O(1)$ time per candidate v_i, thus in $O(n^2)$ total time.

2.1.2 Correctness of the approach

The correctness proof is based upon the following claim.

Claim. *The (finite) values of $f(j,k)$ tabulated in solving the recursion are the lengths of the (existing) solutions to the corresponding subproblems SubProblem(v_1, v_j, v_k).*

This claim follows by induction on the index k, for $1 < k \leq m$. First, the claim holds trivially for $k = 2$, by our definition of $f(1,k)$. Next, for $k \geq 3$, the induction hypothesis is: assume that recursively computed (finite) $f(j', k')$ is the minimum length of a chain solving SubProblem$(v_1, v_{j'}, v_{k'})$ for all values of $1 \leq j' < k' < k$. Again observe that the claim holds for $j = 1$ by our definition of $f(1,k)$, and consider SubProblem(v_1, v_j, v_k), for $1 < j < k$; let $(v_1, v_{j_1}, v_{j_2}, \ldots, v_{j_N}, v_j)$ be a

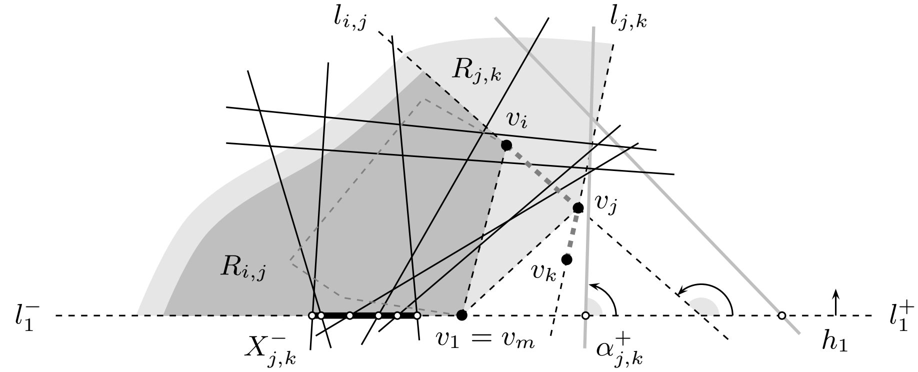

Fig. 7: Selecting the vertex v_i in the recursion to solve SubProblem(v_1, v_j, v_k): Both lines $l_{i,j}$ and $l_{j,k}$ intersect l_1^+. Since $\alpha_{j,k}^+$ is not larger than the angle that $l_{i,j}$ forms with l_1^+, not all lines in $\mathcal{L}_{j,k} \setminus \mathcal{L}_{i,j}$ intersect $v_i v_j$; the relevant lines are marked gray.

minimum-length chain solving the subproblem. The subchain $(v_1, v_{j_1}, \ldots, v_{j_N})$ must be optimal (and finite-length) for the SubProblem(v_1, v_{j_N}, v_j) — otherwise, if a shorter feasible chain existed, we could improve upon the chain $(v_1, v_{j_1}, \ldots, v_{j_N}, v_j)$. And since $1 \le j_N < j < k$, the value $f(j_N, j)$ for SubProblem(v_1, v_{j_N}, v_j) has been correctly computed by the induction hypothesis, and thus $f(j,k) = f(j_N, j) + |\pi(v_{j_N}, v_j)|$.

In the recursion, we must have considered the choice $v_i = v_{j_N}$, $j_N < j$, the next-to-last vertex on the optimal chain, and v_i must have satisfied conditions (i-ii) in our Bellman recursion. Consequently, the hypothesis continues to hold, and so the claim. Thereby we may conclude with the following theorem.

Theorem 1. *Given a set $\mathcal{L}$ of n non-parallel lines in the plane, the watchman route problem for $\mathcal{L}$ can be solved in $O(n^8)$ time.*

Proof. We have given the algorithm and proved correctness above. It remains to argue that the algorithm can be implemented in $O(n^8)$ time.

Handling the cases $h = 1, 2$ requires $O(n^4)$ time. For $h \ge 3$, for each of the $O(n^2)$ choices of v_1, we sort the vertices in $O(n^2 \log n)$ time, and then we evaluate $f(j,k)$ for $O(n^4)$ pairs (v_j, v_k). Each evaluation requires consideration of $O(n^2)$ choices of v_i in evaluating the recursion, with conditions (i)-(ii) determining $I_{j,k}$ checkable in time $O(1)$ per v_i, after $O(n^7)$-time preprocessing for computing the lengths of all paths $\pi(v_i, v_j)$, determining eligible pairs, the interval $X_{j,k}^-$, the set $\mathcal{L}_{j,k}^+$, and angle $\alpha_{j,k}^+$, as described above. Thus, the overall running time is $O(n^8)$. □

A careful examination shows that our approach works in a more general case, when rather than requiring that all lines in $\mathcal{L}$ be visited, we only require that a subset $\mathcal{L}' \subseteq \mathcal{L}$ of lines be visited by a watchman route contained in $\mathcal{A}(\mathcal{L})$. We only need to redefine appropriately the minimum convex hull problem and some of the definitions in the specification of our dynamic programming algorithm, so that only lines in $\mathcal{L}'$ are to be visited. We omit the straightforward details and conclude with the following.

Theorem 2. *Given a set $\mathcal{L}$ of n non-parallel lines in the plane and a subset $\mathcal{L}' \subseteq \mathcal{L}$, the watchman route problem for $\mathcal{L}'$ can be solved in $O(n^8)$ time.*

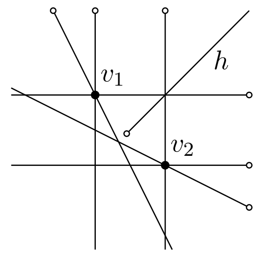

Fig. 8: The line segment v_1v_2 intersects all half-lines, but the shortest tour $\pi(v_1, v_2) \cup \pi(v_2, v_1)$ visiting v_1 and v_2 misses the half-line h. The endpoints of the half-lines are marked with small empty circles.

Remark. A natural question is whether the above approach can be extended to the watchman route problem for half-lines in the plane. Indeed, it appears that this variant is also polynomially tractable [28]. An example depicted in Fig. 8 shows that we cannot simply reduce this watchman route variation to the problem of determining the minimum-length cyclic sequence $(v_1, \ldots, v_h, v_1)$ such that the convex polygon $v_1, \ldots, v_h$ intersects all half-lines (and then to concatenate the shortest paths $\pi(v_1, v_2), \ldots, \pi(v_{h-1}, v_h)$, and $\pi(v_h, v_1)$): in this example, while the line segment v_1v_2 intersects all half-lines, the shortest tour $\pi(v_1, v_2) \cup \pi(v_2, v_1)$ visiting v_1 and v_2 misses the half-line h.

In the next subsection we establish the NP-hardness of the watchman route problem for lines in $\mathbb{R}^3$, and in Section 3 the NP-hardness of the watchman route problem for line segments in $\mathbb{R}^2$, respectively.

2.2 The watchman route problem for lines in 3D

We relate our Watchman Route Problem for Lines (WRL) to the Geometric Traveling Salesman Problem (GTSP) [16, 31]. The latter can be formulated as a decision problem as follows: *Given a set of n lattice points in the plane, and a positive integer m, does there exists a tour of total length at most m that visits all the points?* GTSP is known to be NP-hard with respect to both the L_1 and the L_2 metric [15, 31], and based upon this result, we obtain the following.

Theorem 3. *The watchman route problem for lines (or line segments) in 3D is NP-hard. The problem remains so even for orthogonal lines (or line segments).*

Proof. Given a set P of n lattice points in a horizontal plane Π, let $\mathcal{V}$ be set of n vertical lines in 3D each incident to a point in P, and let $\mathcal{H}$ be the set of axis-aligned lines in Π determined by P (Fig. 9) that form the Hanan grid [20], i.e., through each point $p \in P$, we include in $\mathcal{H}$ the two lines parallel to the coordinate axes.

Let now $\mathcal{L} = \mathcal{H} \cup \mathcal{V}$ be the set of lines to be visited by a watchman route; observe that $\mathcal{L}$ is connected. It is obvious that an optimal route must lie in the plane Π and is given by an optimal axis-aligned route that visits all points in P. Since GTSP for points in the plane is NP-hard with respect to the L_1 metric, it follows that the watchman route problem for lines in 3D is NP-hard. $\square$

In the next section we consider the watchman route problem for line segments in the plane. We provide a polylogarithmic approximation algorithm for WRS, running in polynomial time. The algorithm also applies to the watchman route problem for lines in 3D.

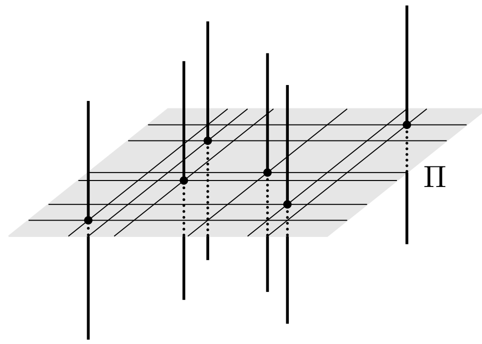

Fig. 9: NP-hardness reduction for lines in 3D.

3 The watchman route problem for segments

In this section, we discuss the watchman route problem for line segments in the plane. In particular, we show that WRS is NP-hard even for axis-parallel segments with a simpler proof than [43, 44]. Then we provide an approximation algorithm with ratio $O(\log^3 n)$ for connected set of segments in any dimension, and show that an approximation algorithm with a constant ratio exists for certain special cases. Finally, we give a fast exact algorithm for WRS in outerplanar grids.

3.1 NP-hardness

Theorem 4. *The watchman route problem for segments in the plane is NP-hard. The problem remains so even for axis-aligned line segments.*

Proof. We adapt the NP-hardness proof of computing a shortest watchman route in a polygon with holes [12]. Given a set P of n lattice points, we construct a set $\mathcal{S}$ of axis-aligned line segments as follows. Let R be the minimal axis-parallel bounding rectangle containing P. First, we construct the Hanan grid induced by the points (Fig. 10): for each point $p \in P$, we add to $\mathcal{S}$ the maximal (in R) horizontal and vertical line segments incident to p. Then, for each point $p \in P$, we add to $\mathcal{S}$

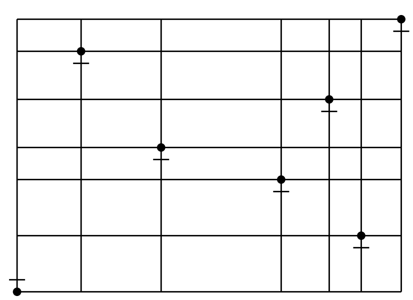

Fig. 10: NP-hardness reduction for line segments in 2D.

a short horizontal line segment $s(p)$ of length $\frac{1}{10n}$ at distance $\frac{1}{20n}$ from p in R. Observe that $s(p)$ can be only visited from the grid segment incident to p. The reduction, hence the NP-hardness of WRS, follows via the following claim.

Claim. *For a positive integer m, there exists a tour of P of length at most m in the L_1 metric if and only if there exists a watchman route for $\mathcal{S}$ of length at most $m + 0.1$.*

Proof of Claim. The direct implication is easy: given a tour of P, convert it into a tour of $\mathcal{S}$ by augmenting it with at most n detours of length $2 \cdot \frac{1}{20n}$ each, as needed. The total cost of the augmentation does not exceed $1/10$, as required.

For the converse implication, given a tour of $\mathcal{S}$ of length at most $m + 0.1$, first convert it into a tour of P by augmenting it with at most n detours of length $2 \cdot \frac{1}{20n}$, as needed. The total cost of the augmentation does not exceed $1/10$, and we have now a tour of P, say $p_1 p_2 \ldots p_n$, of length at most $m + 0.2$. Let π_i denote the path connecting the lattice points p_i to p_{i+1} in this tour, where $p_{n+1} = p_1$. For each $i = 1, \ldots, n$, convert π_i into a path π'_i of length at most $\lfloor |\pi_i| \rfloor$ (take for instance the shortest path in the grid connecting these grid points). By concatenating these paths yields a tour of P of some integer length at most m, as required.

This concludes the proof of Claim and thereby the proof of Theorem 4. □

3.2 Approximation algorithm

Reich and Widmayer [32] introduced the following *group Steiner tree* (a.k.a., *one-of-a-set Steiner tree*) problem. Given an undirected graph $G = (V, E)$ on n vertices and with weighted edges, and k subsets of V called *groups*, find a minimum-weight tree that has at least one vertex from each group. The problem is known to be APX-hard [3], and the current best approximation ratio, $O(\log^2 n \log k)$, originates from the randomized algorithm of Garg et al. [17] as further refined by Fakcharoenphol et al. [14]. The cycle version of this problem, the *generalized traveling salesman problem*, was introduced even earlier by Henry-Labordere [21] and Saskena [33]; clearly, the same approximation ratio, $O(\log^2 n \log k)$, holds.

In order to apply this approximation result to our problem, the graph $G = (V, E)$ is set to the weighted planar graph $G(\mathcal{S})$ (see Section 1), and groups correspond to the sets of intersection points along each of the n input segments; so we have $k = n$ groups. Hence, the $O(\log^2 n \log k)$ approximation algorithm from [14] yields an approximation ratio $O(\log^3 n)$ for WRS; we point out that it applies to any dimension d.

Theorem 5. *There is a randomized approximation algorithm with ratio $O(\log^3 n)$ for computing a shortest watchman route for a connected set of n line segments in $\mathbb{R}^d$, for any fixed dimension d.*

Remark. A deterministic algorithm with a better approximation ratio for the planar case, $O(\log^2 n)$, has been recently proposed by Mitchell [28].

3.3 Light segments

In the special case that each segment $s \in \mathcal{S}$ has at most a constant number of intersection points with other segments, we apply a result of Slavik [35]: the generalized TSP in graphs (called the "errand scheduling problem" by Slavik) has a $\frac{3c}{2}$-approximation algorithm, where c is the maximum size of a group. (Notice that we do not require that at most a constant number of line segments have a point in common). Consequently, we obtain the following result.

Theorem 6. *Suppose that for each line segment $s \in \mathcal{S}$, there are at most c intersection points on s. Then, there is a polynomial time algorithm for the watchman route problem with an approximation factor of $\frac{3c}{2}$.*

3.4 Outerplanar grids

We say that an arrangement $\mathcal{A}(\mathcal{S})$ of line segments is *outerplanar* if each segment in $\mathcal{S}$ has a point on the boundary of the outer face of the arrangement. Following [18, 25], we say that an arrangement $\mathcal{A}(\mathcal{S})$ of line segments is *simple* if all endpoints of the segments in $\mathcal{S}$ lie on the outer face of the arrangement and if each segment $s \in \mathcal{S}$ can be extended by any arbitrarily small $\varepsilon > 0$ in both directions such that its new endpoints still lie on the outer face. Every simple arrangement is outerplanar, but not every outerplanar arrangement is simple.

Recall that an arrangement of axis-aligned segments is called a grid. Thus an *outerplanar grid* (our focus in this section) is an outerplanar arrangement of axis-aligned segments; see Fig. 11(a) for an example. Guarding problems on *simple grids*, have been studied in [18, 25].

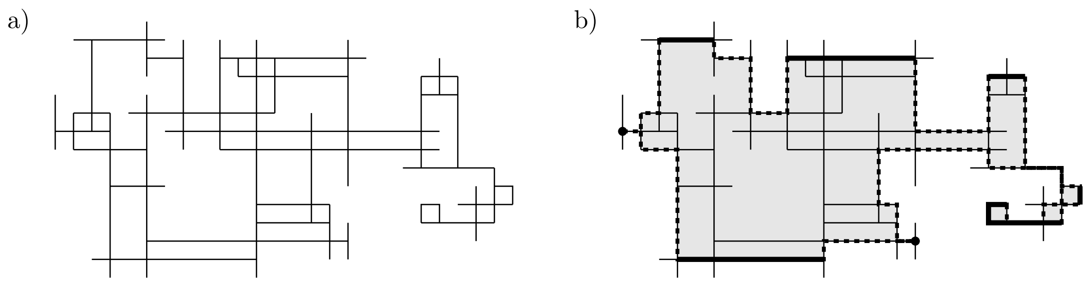

Fig. 11: (a) An outerplanar grid $\mathcal{G}$: every grid segment has a point on the outer face of the planar subdivision formed by the grid. (b) The boundary of the enclosing polygon $P(\mathcal{G})$ is marked with dashed lines. Essential edges of $P(\mathcal{G})$ are marked with bold solid lines/dots.

Given an outerplanar grid $\mathcal{G}$, let $P(\mathcal{G})$ be the minimal orthogonal polygon (with respect to inclusion) enclosing all vertices (segment intersections) and bounded faces of $\mathcal{G}$ and such that $\partial P(\mathcal{G}) \subseteq \mathcal{G}$, see Fig. 11(b); $P(\mathcal{G})$ is called the *enclosing polygon of* $\mathcal{G}$. Observe that the edges of the polygon $P(\mathcal{G})$ are alternating horizontal and vertical (some edges can have 0 length, however, consecutive degenerate edges are excluded). An edge e of $P(\mathcal{G})$ is *essential* if both its endpoints are convex vertices in $P(\mathcal{G})$; in the degenerate case, e can be a point; see again Fig. 11(b). Obviously, all essential edges must be visited by a watchman tour.

We argue that the perimeter of $P(\mathcal{G})$ is a shortest watchman route for the grid. This follows from the following two claims.

Claim 1. *If there is a route visiting all essential edges, then there is a non-self-crossing route of the same length visiting them in boundary order (i.e., their order along $\partial P(\mathcal{G})$).*

Proof. The claim follows by the same arguments as those in the proof of Lemma 1 in [18]. Consider any watchman route that visits the essential edges in an order other than the order in which they appear in the clockwise scan of $\partial P(\mathcal{G})$. Then there must exist at least four essential edges, not necessarily consecutive, that are visited in the order depicted in Fig. 12, i.e., the route is self-crossing. Now, we can replace this route with one that visits these edges in the required order, thus decreasing the number of self-crossings, and whose length is the same as the initial one. Clearly, the new route visits the same set of essential edges.

Applying the same argument as many times as needed, decreasing each time the number of 4-tuples that are out of order (or equivalently, the number of self-crossings), we eventually obtain a non-self-crossing route whose length is the same as the initial one, as required. ☐

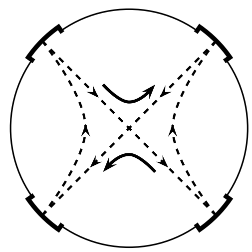

Fig. 12: There exists a non-crossing shortest route visiting all the essential edges.

Claim 2. *The perimeter of $P(\mathcal{G})$ is a shortest route that visits all the essential edges in the same order as they appear on the boundary of $P(\mathcal{G})$.*

Proof. Assume that $P(\mathcal{G})$ has m essential edges $e_1, e_2, \ldots, e_m$, and let $\mathcal{R}$ be the shortest route that visits all the essential edges in the same (clockwise) order as they appear in the boundary ∂P of $P(\mathcal{G})$. The route $\mathcal{R}$ can be decomposed into m paths Π_i connecting points $p_i \in e_i$ and $p_{i+1} \in e_{i+1}$, for $i = 1, \ldots, m$, respectively, where $p_{m+1} = p_1$ and $e_{m+1} = e_1$. Observe that Π_i is a shortest path connecting p_i and p_{i+1}, $i = 1, \ldots, m$, since otherwise, $\mathcal{R}$ can be shortened, which contradicts its minimality.

It suffices to show that if $\Pi_1 \not\subset \partial P$ then Π_1 can be replaced by the boundary path $\Pi'_1 = \partial P(p_1, p_2) \subset \partial P$ from p_1 to p_2, in clockwise order; the same argument can be the applied for each of the paths Π_i, $i = 2, \ldots, m$.

So suppose that $\Pi_1 \not\subset \partial P$. Let $q' \in \Pi_1$ be the first point where Π_1 leaves $\partial P(p_1, p_2)$, and let $q'' \in \Pi_1$ be the first point after q' where Π_1 again enters ∂P; see Fig. 13 for an illustration. Without loss of generality assume that q' belongs to a horizontal edge e' and q'' to a vertical edge

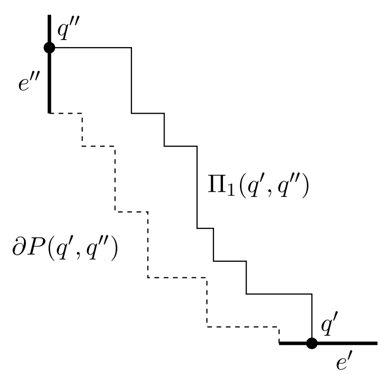

Fig. 13: Since there are no essential edges between e' and e'', and Π_1 is the shortest path connecting p_1 and p_2, $\Pi_1(q', q'')$ can be replaced with $\partial P(q', q'')$.

e''; if q' (resp. q'') is a vertex incident to two edges e'_1 and e'_2 (resp. e''_1 and e''_2) in clockwise order, then e' (resp. e'') is set to e'_2 (resp. e''_1). Then, since there are no essential edges between e_1 and e_2, and so between e' and e'', and Π_1 is a shortest path connecting p_1 and p_2, the subpath $\partial P(q', q'')$ forms a "staircase"; that is, while traversing $\partial P(q', q'')$ from q' to q'', one moves either up or to the left, always starting (resp. ending) with a vertical (resp. horizontal) segment. Consequently, $\partial P(q', q'')$ is also a shortest path connecting q' and q'', and hence $\Pi_1(q', q'')$ can be replaced with $\partial P(q', q'')$. By applying similar replacements to all non-boundary parts of Π_1, we obtain the path Π'_1 connecting p_1 and p_2 such that $\Pi'_1 \subset \partial P$, as required. □

By the definition of outerplanar grids, any grid segment has an intersection point on the perimeter of $P(\mathcal{G})$, thus by traversing the perimeter, all segments are visited; i.e., the perimeter is a shortest watchman route of the whole grid. The enclosing polygon $P(\mathcal{G})$ is readily obtained from the face at infinity in the arrangement of the n segments. For axis-aligned segments, this can can be computed in $O(n \log n)$ time by ray-shooting [19, 22].

Theorem 7. *The watchman route problem for an outerplanar grid $\mathcal{G}$ with n segments can be solved in $O(n \log n)$ time.*

One can ask whether the same approach can be applied to outerplanar segment arrangements in which segments are not necessarily axis-aligned. The examples in Fig. 14 show that the perimeter of the enclosing polygon $P(\mathcal{S})$ of a connected set $\mathcal{S}$ of segments[2] may fail to be a shortest watchman route, and moreover, that a shortest route visiting the essential edges of $P(\mathcal{S})$ may fail to visit all segments.

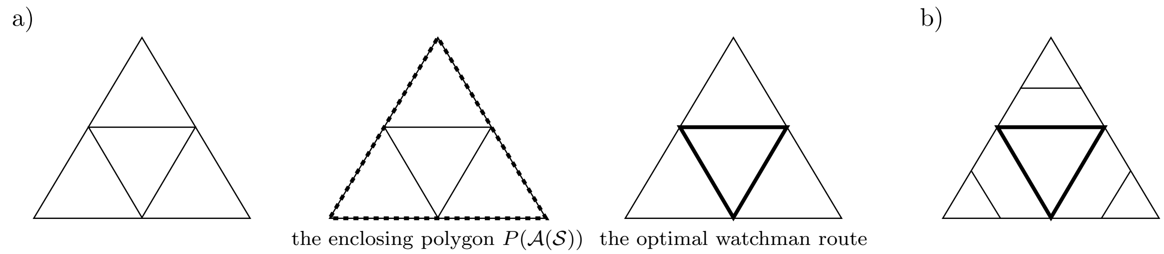

Fig. 14: (a) A outerplanar segment arrangement $\mathcal{A}(\mathcal{S})$, its enclosing polygon $P(\mathcal{S})$, and the shortest watchman route. (b) The shortest route visiting all essential edges of $P(\mathcal{S})$ may fail to be a watchman route.

4 Conclusion

It should be possible to extend our methods in Sections 2 and 3 to the subclass of outerplanar arrangements; in particular, we believe that the following statement holds: Given a connected set $\mathcal{S}$ of n line segments that forms an *simple* arrangement, and a subset $\mathcal{S}' \subseteq \mathcal{S}$, the watchman route problem for $\mathcal{S}'$ can be solved in polynomial time. We leave this as a future direction.

We conclude with a few other open problems concerning watchman routes in line/segment arrangements:

(i) Can the running time of our algorithm for the watchman route problem for lines in the plane be improved?

(ii) Can the $O(\log^2 n)$ approximation factor for segments in the plane (from [28]) be improved?

(iii) What is the complexity of the watchman route problem for planes in 3D?

(iv) How fast can one solve the watchman *path* problem for lines in the plane? Given a set $\mathcal{L}$ of non-parallel lines in the plane, the problem is to find a shortest curve (watchman path) contained in the union of the lines in $\mathcal{L}$ such that every line is visited by the path. Using ideas from [5] for the watchman path problem in polygons, we believe that our methods can be adapted to obtain a polynomial-time algorithm for the path version of our problem.

[2] The enclosing polygon $P(\mathcal{S})$ for a set $\mathcal{S}$ of segments is defined analogously to the enclosing polygon $P(\mathcal{G})$ for a grid $\mathcal{G}$.

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