Online Unit Covering in Euclidean Space*

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Abstract

We revisit the online Unit Covering problem in higher dimensions: Given a set of n points in $\mathbb{R}^d$, that arrive one by one, cover the points by balls of unit radius, so as to minimize the number of balls used. In this paper, we work in $\mathbb{R}^d$ using the Euclidean distance.

(I) We give an online deterministic algorithm with competitive ratio $O(1.321^d)$, thereby improving on the previous record, $O(2^d d \log d)$, due to Charikar et al. (2004), by an exponential factor. In particular, the competitive ratios are 5 in the plane and 12 in 3-space (the previous ratios were 7 and 21, respectively). For $d = 3$, the ratio of our online algorithm matches the ratio of the current best offline algorithm for the same problem due to Biniaz et al. (2017), which is remarkable (and rather unusual).

(II) We show that the competitive ratio of every deterministic online algorithm for Unit Covering in $\mathbb{R}^d$ under the L_2 norm is at least $d + 1$ for every $d \geq 1$. This greatly improves upon the previous best lower bound, $\Omega(\log d/ \log\log\log d)$, due to Charikar et al. (2004).

(III) We generalize the above result to Unit Covering in $\mathbb{R}^d$ under the L_C norm, where C is a centrally symmetric convex body, via the illumination number.

(IV) We obtain lower bounds of 4 and 5 for the competitive ratio of any deterministic algorithm for online Unit Covering in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively; the previous best lower bounds were 3 for both cases.

(V) When the input points are from the square or hexagonal lattice in $\mathbb{R}^2$, we give deterministic online algorithms for Unit Covering with an optimal competitive ratio of 3. For the cubic lattice in $\mathbb{R}^3$, we give a deterministic online algorithm with a competitive ratio of 5.

Keywords: online algorithm, unit covering, unit clustering, competitive ratio, illumination number, Newton number.

1 Introduction

Covering and clustering are fundamental problems in the theory of algorithms, computational geometry, optimization, and other areas. They arise in a wide range of applications, such as facility location, information retrieval, robotics, and wireless networks. While these problems have been studied in an offline setting for decades, they have been considered only recently in a more dynamic (and thereby realistic) setting. Here we study such problems in a high-dimensional Euclidean space and mostly in the L_2 norm. We first formulate them in the classic *offline* setting.

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Problem 1. k-Center. *Given a set of n points in $\mathbb{R}^d$ and a positive integer k, cover the set by k congruent balls centered at the points so that the diameter of the balls is minimized.*

The following two problems are dual to Problem 1.

Problem 2. Unit Covering. *Given a set of n points in $\mathbb{R}^d$, cover the set by balls of unit diameter so that the number of balls is minimized.*

Problem 3. Unit Clustering. *Given a set of n points in $\mathbb{R}^d$, partition the set into clusters of diameter at most one so that the number of clusters is minimized.*

Problems 1 and 2 are easily solved in polynomial time for points on the line, i.e., for $d = 1$; but both problems become NP-hard already in Euclidean plane [20, 33]. Factor 2 approximations are known for k-Center in any metric space (and so for any dimension) [19, 22]; see also [36, Ch. 2], while polynomial-time approximation schemes are known for Unit Covering for any fixed dimension [24]. However, these algorithms are notoriously inefficient and thereby impractical; see also [4] for a summary of such results and different time vs. ratio trade-offs.

Problems 2 and 3 are identical in the offline setting: indeed, one can go from clusters to balls in a straightforward way; and conversely, one can assign points that are covered multiple times in an arbitrary fashion to unique balls. In this paper we focus on the second problem, namely *online* Unit Covering; we however point out key differences between this problem and online Unit Clustering.

The performance of an online algorithm ALG is measured by comparing it to an optimal offline algorithm OPT using the standard notion of competitive ratio [5, Ch. 1]. The competitive ratio of ALG is defined as $\sup_\sigma \frac{\mathsf{ALG}(\sigma)}{\mathsf{OPT}(\sigma)}$, where the supremum is taken over all input sequences σ, $\mathsf{OPT}(\sigma)$ is the cost of an optimal offline algorithm for σ, and $\mathsf{ALG}(\sigma)$ denotes the cost of the solution produced by ALG for this input. For randomized algorithms, $\mathsf{ALG}(\sigma)$ is replaced by the expectation $E[\mathsf{ALG}(\sigma)]$, and the competitive ratio of ALG is $\sup_\sigma \frac{E[\mathsf{ALG}(\sigma)]}{\mathsf{OPT}(\sigma)}$. If there is no danger of confusion, we use ALG to refer to an algorithm or the cost of its solution, as needed.

Throughout this paper, all our upper bounds and lower bounds are deterministic, and all lower bounds are given by adaptive online adversaries. An *adaptive online adversary* constructs the next point online, based on the previous actions of the algorithm.

Charikar et al. [11, Sec. 6] studied the online version of Unit Covering (under the name of "Dual Clustering"). The points arrive one by one and each point needs to be assigned to a new or to an existing unit ball upon arrival; the L_2 norm is used in $\mathbb{R}^d$, $d \in \mathbb{N}$. The location of each new ball is fixed as soon as it is placed. The authors provided a deterministic algorithm of competitive ratio $O(2^d d \log d)$ and gave a lower bound of $\Omega(\log d / \log\log\log d)$ on the competitive ratio of any deterministic algorithm for this problem. For $d = 1$ a tight bound of 2 is folklore; for $d = 2$ the best known upper and lower bounds on the competitive ratio are 7 and 3, respectively, as implied by the results in [11][1].

The online Unit Clustering problem was introduced by Chan and Zarrabi-Zadeh [10] in 2006. While the input and the objective of this problem are identical to those for Unit Covering, Unit Clustering is more flexible in that the algorithm is not required to produce unit balls at any time, but rather the smallest enclosing ball of each cluster should have diameter *at most* 1; furthermore, a ball may change (grow or shift) in time. In regard to their *online* versions, it is worth emphasizing two properties shared with Unit Covering: (i) a point assigned to a cluster must remain in that

[1] Charikar et al. [11] claim (on p. 1435) that a lower bound of 4 for $d = 2$ under the L_2 norm follows from their Theorem 6.2; but this claim appears unjustified; only a lower bound of 3 is implied. Unfortunately, this misinformation has been carried over also by [10] and [14].

cluster; and (ii) two distinct clusters cannot merge into one cluster, i.e., the clusters maintain their identities. The authors showed that several standard approaches for Unit Clustering, namely the deterministic algorithms `Centered`, `Grid`, and `Greedy`, all have competitive ratio at most 2 for points on the line ($d = 1$). Moreover, the first two algorithms are applicable for Unit Covering, with a competitive ratio at most 2 for $d = 1$, as well. These algorithms naturally extend to any higher dimension (including `Grid` provided the L_∞ norm is used).

> `Algorithm Centered.` For each new point p, if p is covered by an existing unit ball, do nothing; otherwise place a new unit ball centered at p.

> `Algorithm Grid.` Build a uniform grid in $\mathbb{R}^d$ where cells are unit cubes of the form $\prod_{j=1}^{d} [i_j, i_j + 1)$, where $i_j \in \mathbb{Z}$ for $j = 1, \ldots, d$. For each new point p, if the grid cell containing p is nonempty, put p in the corresponding cluster; otherwise open a new cluster for the grid cell and put p in it.

Since in $\mathbb{R}^d$ each cluster of OPT can be split into at most 2^d grid-cell clusters created by the algorithm, its competitive ratio is at most 2^d, and this analysis is tight for the L_∞ norm. It is worth noting that there is no direct analogue of this algorithm under the L_2 norm.

Some (easy) remarks are in order. Any lower bound on the competitive ratio of an online algorithm for Unit Clustering applies to the competitive ratio of the same type of algorithm for Unit Covering. Conversely, any upper bound on the competitive ratio of an online algorithm for Unit Covering yields an upper bound on the competitive ratio of the same type of algorithm for Unit Clustering.

Related work. Unit Covering is a variant of Set Cover. Alon et al. [1] gave a deterministic online algorithm of competitive ratio $O(\log m \log n)$ for this problem, where n is the size of the ground set and m is the number of sets in the family. Buchbinder and Naor [9] obtained sharper results under the assumption that every element appears in at most Δ sets.

Chan and Zarrabi-Zadeh [10] showed that no online algorithm (deterministic or randomized) for Unit Covering can have a competitive ratio better than 2 in one dimension ($d = 1$). They also showed that it is possible to get better results for Unit Clustering than for Unit Covering. Specifically, they developed the first algorithm with competitive ratio below 2 for $d = 1$, namely a randomized algorithm with competitive ratio 15/8. This fact has been confirmed by subsequent algorithms designed for this problem; the current best ratio 5/3, for $d = 1$, is due to Ehmsen and Larsen [17], and this gives a ratio of $2^d \cdot \frac{5}{6}$ for every $d \geq 2$ (the L_∞ norm is used); their algorithm is deterministic. The appropriate "lifting" technique has been laid out in [10, 39]. From the other direction, the lower bound for deterministic algorithms has evolved from 3/2 in [10] to 8/5 in [18], and then to 13/8 in [27].

Answering a question of Epstein and van Stee [18], Dumitrescu and Tóth [14] showed that the competitive ratio of any online algorithm (deterministic or randomized) for Unit Clustering in $\mathbb{R}^d$ under the L_∞ norm must depend on the dimension d; in particular, it is $\Omega(d)$ for every $d \geq 2$.

Recently, Liaw et al. [29] studied Unit Covering in the streaming model, where n points arrive one by one, and the algorithm needs to report the minimum *number* of unit disks that cover all points using $O(\log n)$ memory. The authors provide constant-factor randomized approximations algorithms under the L_∞ and L_2 norms in $\mathbb{R}^d$, and show that constant factor-approximations are the best possible in the streaming model. They maintain the number of covering disks in several possible solutions (shifting method), and output the minimum; as such, these algorithms do not apply to the online setting.

Liao and Hu [28] gave a PTAS for a related disk cover problem (another variant of Set Cover): given a set of m disks of arbitrary radii and a set P of n points in $\mathbb{R}^2$, find a minimum-cardinality subset of disks that jointly cover P; see also [34, Corollary 1.1].

Online algorithms for Unit Clustering and Unit Covering are surveyed in [12, 13]. An experimental comparison of existing algorithms for Unit Covering appears in [21].

Our results.

(i) We show that the competitive ratio of `Algorithm Centered` for online Unit Covering in $\mathbb{R}^d$, $d \in \mathbb{N}$, under the L_2 norm is bounded by the Newton number of the Euclidean ball in the same dimension. In particular, it follows that this ratio is $O(1.321^d)$ (Theorem 1 in Section 2). This greatly improves on the ratio of the previous best algorithm due to Charikar et al. [11]. The competitive ratio of their algorithm is at most $f(d) = O(2^d d \log d)$, where $f(d)$ is the number of unit balls needed to cover a ball of radius 2 (i.e., the doubling constant). By a volume argument, $f(d)$ is at least 2^d. In particular $f(2) = 7$ and $f(3) = 21$ [38]; see also [4]. The competitive ratios of our algorithm are 5 in the plane and 12 in 3-space, improving the earlier ratios of 7 and 21, respectively.

(ii) We show that the competitive ratio of every deterministic online algorithm for Unit Covering in $\mathbb{R}^d$ under the L_2 norm is at least $d + 1$ for every $d \geq 1$ (Theorem 3 in Section 3). This greatly improves the previous best lower bound, $\Omega(\log d / \log\log\log d)$, due to Charikar et al. [11].

(iii) We generalize the above result to Unit Covering in $\mathbb{R}^d$ under the L_C norm, where C is a centrally symmetric convex body, via the illumination number (Theorem 4 in Section 3.3).

(iv) We obtain lower bounds of 4 and 5 for the competitive ratio of any deterministic algorithm for Unit Covering in $\mathbb{R}^2$ and respectively $\mathbb{R}^3$ (Theorems 2 and 3 in Section 3). The previous best lower bounds were both 3.

(v) For input point sequences that are subsets of the infinite square or hexagonal lattices, we give deterministic online algorithms for Unit Covering with an optimal competitive ratio of 3 (Theorems 5 and 6 in Section 4). For the cubic lattice in $\mathbb{R}^3$, we give a deterministic online algorithm with a competitive ratio of 5 (Theorem 7 in Section 4).

Notation and terminology. For two points $p, q \in \mathbb{R}^d$, let $d(p, q)$ denote the Euclidean distance between them. Throughout this paper the L_2-norm is used. The closed ball of radius r in $\mathbb{R}^d$ centered at point $z = (z_1, \ldots, z_d)$ is

$$B_d(z, r) = \{x \in \mathbb{R}^d \mid d(z, x) \leq r\} = \{(x_1, \ldots, x_d) \mid \sum_{i=1}^{d} (x_i - z_i)^2 \leq r^2\}.$$

A *unit ball* is a ball of unit radius in $\mathbb{R}^d$. The Unit Covering problem is to cover a set of points in $\mathbb{R}^d$ by a minimum number of unit balls.

The *unit sphere* is the surface of the d-dimensional unit ball centered at the origin $\mathbf{0}$, namely, the set of points $\mathbb{S}^{d-1} \subset B_d(\mathbf{0}, 1)$ for which equality holds: $\sum_{i=1}^{d} x_i^2 = 1$. A *spherical cap* $C(\alpha)$ of angular radius $\alpha \leq \pi$ and center P on $\mathbb{S}^{d-1}$ is the set of points Q in $\mathbb{S}^{d-1}$ for which $\angle P\mathbf{0}Q \leq \alpha$; see [35]. A *unit cube* is an axis-aligned cube of unit side-length.

2 Analysis of `Algorithm Centered` for online unit covering in Euclidean d-space

For a convex body $C \subset \mathbb{R}^d$, the *Newton number* (a.k.a. *kissing number*) of C is the maximum number of nonoverlapping congruent copies of C that can be arranged around C so that each of them is touching C [8, Sec. 2.4]. Some values $N(B_d)$, where $B_d = B_d(\mathbf{0}, 1)$, are known exactly for small d, while for most dimensions d we only have estimates. For instance, it is easy to see that $N(B_2) = 6$, and it is known that $N(B_3) = 12$ and $N(B_4) = 24$. The problem of estimating $N(B_d)$ in higher dimensions is closely related to the problem of determining the densest sphere packing and the knowledge in this area is largely incomplete with large gaps between lower and upper bounds; see [8, Sec. 2.4] and the references therein; in particular, many upper and lower estimates up to $d = 128$ are given in [7] and [16]. In this section, we prove the following theorem.

Theorem 1. *Let $\varrho(d)$ be the competitive ratio of* `Algorithm Centered` *in $\mathbb{R}^d$ (when using the L_2 norm). Then $\varrho(2) = N(B_2) - 1 = 5$, $\varrho(3) = N(B_3) = 12$, and $\varrho(d) \leq N(B_d)$ for every $d \geq 4$. In particular, $\varrho(d) = O(1.321^d)$.*

A key fact for proving the theorem is the following easy lemma.

Lemma 1. *Let B be a unit ball centered at o, that is part of OPT. Let $p, q \in B$ be any two points in B presented to the online algorithm that forced the algorithm to place new balls centered at p and q; refer to Fig. 1. Then $\angle poq > \pi/3$.*

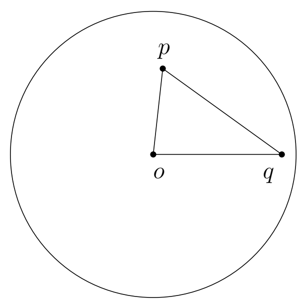

Figure 1: Illustration for Lemma 1.

Proof. Assume for contradiction that $\alpha = \angle poq \leq \pi/3$. Assume also, as we may, that p arrives before q. Since $q \notin B(p)$, we have $|pq| > 1$. Consider the triangle Δpoq; we may further assume that $\angle opq \geq \angle oqp$ (if we have the opposite inequality, the argument is symmetric). In particular, we have $\angle opq \geq \pi/3$. Since $\angle poq \leq \pi/3$ and $\angle opq \geq \pi/3$, the law of sines yields $|oq| \geq |pq| > 1$. However, this contradicts the fact that q is contained in B, and the proof is complete. □

Corollary 1. *Let B be a unit ball centered at o, that is part of OPT. For every point $p \in B$ presented to the online algorithm that forced the algorithm to place a new ball centered at p, let $\Psi(p)$ denote the cone with apex at o, axis $\overrightarrow{op}$, and angle $\pi/6$ around $\overrightarrow{op}$. Then the cones $\Psi(p)$ are pairwise disjoint in B; hence the corresponding caps on the surface of B are also nonoverlapping.*

Proof of Theorem 1. For every unit ball B of OPT we bound from above the number of unit balls placed by `Algorithm Centered` whose center lies in B. Suppose this number is at most A (for every ball in OPT). Since the center of every unit ball placed by the algorithm is a point of the set and all points in the set are covered by balls in OPT, it follows that the competitive ratio of `Algorithm Centered` is at most A.

By Corollary 1 we are interested in the maximum number $A(\alpha)$ of nonoverlapping caps $C(\alpha)$ that can be placed on $\mathbb{S}^{d-1}$, for $\alpha = \pi/6$. This is precisely the maximum number of nonoverlapping balls that can touch a fixed unit ball externally, which is the Newton number $N(B_d)$ in dimension d.

For $d = 2$ we gain 1 in the bound due to the fact that the inequality in Lemma 1 is strict and we are dealing with the unit circle; the five vertices of a regular pentagon inscribed in a unit circle make a tight example with ratio 5; note that the minimum pairwise distance between points is $2\sin(\pi/5) > 1$, and so the algorithm places a new ball for each point. For $d = 3$ the twelve vertices of a regular icosahedron inscribed in a unit sphere make a tight example with ratio 12; since the minimum pairwise distance between points is $(\sin(2\pi/5))^{-1} > 1$, the same observation applies. □

Bounds on the Newton number of the ball. A classic formula established by Rankin [35] yields

$$N(B_d) \leq \sqrt{\frac{\pi}{8}}\, d^{3/2}\, 2^{d/2}(1 + o(1)). \tag{1}$$

More recently, Kabatiansky and Levenshtein [26] have established a sharper upper bound

$$N(B_d) \leq 2^{0.401d(1+o(1))}. \tag{2}$$

In particular, $N(B_d) = O(1.321^d)$. It is worth noting that the current lower and upper bounds on the Newton number are far apart. The current best lower bound, due to Jenssen et al. [25] (see also [37]), is

$$N(B_d) = \Omega\left(d^{3/2} \cdot \left(\frac{2}{\sqrt{3}}\right)^d\right). \tag{3}$$

In particular, $N(B_d) = \Omega(1.154^d)$.

3 Lower bounds on the competitive ratio for online unit covering in Euclidean d-space

Theorem 3, that we prove in this section, greatly improves the previous best lower bound on the competitive ratio of a deterministic algorithm, $\Omega(\log d/ \log\log\log d)$, due to Charikar et al. [11].

Previous lower bounds for $d = 2$ and 3. To clarify matters, we briefly summarize the calculation leading to the previous best lower bounds on the competitive ratio. Charikar et al. [11, p. 1435] claim that a lower bound of 4 for $d = 2$ under the L_2 norm follows from their Theorem 6.2; but this claim appears unjustified; only a lower bound of 3 is implied. The proof uses a volume argument. For a given d, the parameters R_t are iteratively computed for $t = 1, 2, \ldots$ by using the recurrence relation

$$R_{t+1} = \frac{R_t + t^{1/d}}{2}, \text{ where } R_1 = 0. \tag{4}$$

The lower bound on the competitive ratio of any deterministic algorithm given by the argument is the largest t for which $R_t \leq 1$. The values obtained for R_t, for $t = 1, 2, \ldots$ and $d = 2, 3$ are listed in Table 1; as such, both lower bounds are equal to 3.

d	R_1	R_2	R_3	R_4
2	0	0.5	0.957...	1.344...
3	0	0.5	0.879...	1.161...

Table 1: Values R_t, for $t = 1, 2, 3, 4$.

3.1 A new lower bound in the plane

In this section, we deduce an improved lower bound of 4 (an alternative proof will be provided by Theorem 3).

Theorem 2. *The competitive ratio of any deterministic online algorithm for* Unit Covering *in the Euclidean plane (under the L_2 norm) is at least* 4.

Proof. Consider a deterministic online algorithm ALG. We present an input instance σ for ALG and show that the solution ALG(σ) is at least 4 times OPT(σ). Our proof works like a two-player game, played by Alice and Bob. Here, Alice is presenting points to Bob, one at a time. Bob (who plays the role of the algorithm) makes the decision whether to place a new disk or not. If a new disk is required, Bob decides where to place it. Alice tries to force Bob to place as many new disks as possible by presenting the points in a smart way. Bob tries to place new disks in a way such that they may cover other points presented by Alice in the future, thereby reducing the need of placing new disks quite often.

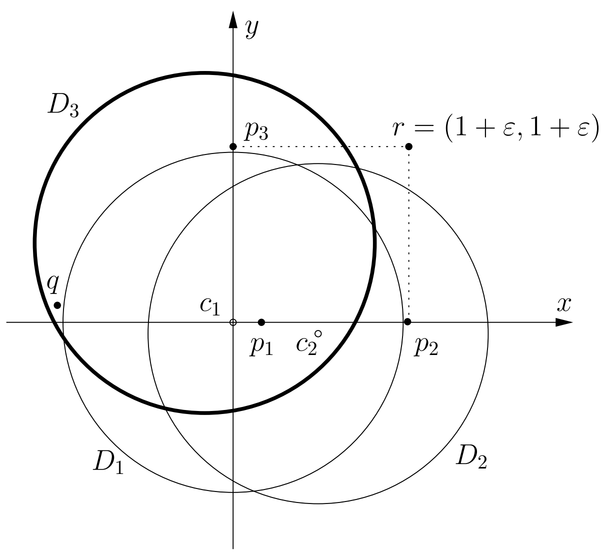

Figure 2: A lower bound of 4 on the competitive ratio in the plane. The figure illustrates the case $p_4 = r$.

The center of a disk D_i is denoted by c_i, $i = 1, 2, \ldots$ The coordinates of the points will depend on a parameter $\varepsilon > 0$; a sufficiently small $\varepsilon \leq 0.01$ is chosen so that the inequalities appearing in the proof hold. First, point p_1 arrives and the algorithm places disk D_1 to cover it. Without loss of generality, assume that $c_1 = (0, 0)$ and $p_1 = (x, 0)$, where $0 \leq x \leq 1$. The second point presented is $p_2 = (1 + \varepsilon^2, 0)$ and, since $p_2 \notin D_1$, a second disk D_2 is needed to cover it. By symmetry, we may assume that $y(c_2) \leq 0$; that is, $c_2 = (x_2, y_2)$, where $x_2 \geq \varepsilon^2$ and $y_2 \leq 0$. The third point presented is $p_3 = (0, 1 + \varepsilon)$, and neither D_1 nor D_2 covers it; thus a new disk, D_3, is required to cover p_3.

Consider two other candidate points, $q = (-1+\varepsilon, \sqrt{2\varepsilon})$ and $r = (1+\varepsilon, 1+\varepsilon)$. Since

$$|qc_1|^2 = (-1+\varepsilon)^2 + 2\varepsilon = 1 + \varepsilon^2 - 2\varepsilon + 2\varepsilon = 1 + \varepsilon^2 > 1,$$

q is not covered by D_1; and clearly r is not covered by D_1. Since

$$|qc_2|^2 \geq (x_2 + 1 - \varepsilon)^2 + 2\varepsilon \geq (1 - \varepsilon + \varepsilon^2)^2 + 2\varepsilon = 1 + 3\varepsilon^2 + O(\varepsilon^3) > 1,$$

q is not covered by D_2; and clearly r is not covered by D_2. Note also that the D_3 cannot cover both q and r, since their distance is close to $\sqrt{5} > 2$. We now specify p_4, the fourth point presented to the algorithm. If q is covered by D_3, let $p_4 = r$, otherwise let $p_4 = q$. In either case, a fourth disk, D_4, is required to cover p_4.

To conclude the proof, we verify that p_1, p_2, p_3, and p_4 can be covered by a unit disk.

Case 1: $p_4 = r$. It is easily seen that p_1, p_2, p_3, and p_4 can be covered by the unit disk D centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$; indeed, all four points are close to the boundary of the unit square $[0,1]^2$.

Case 2: $p_4 = q$. Consider the unit disk D centered at the midpoint c of qp_2. We have

$$|qp_2|^2 = (2 - \varepsilon + \varepsilon^2)^2 + 2\varepsilon = 4 - 2\varepsilon + O(\varepsilon^2) < 4.$$

It follows that D covers p_2 and p_4. Note that

$$c = \left(\frac{\varepsilon + \varepsilon^2}{2}, \sqrt{\frac{\varepsilon}{2}}\right).$$

We next check the containment of p_1 and p_3.

$$|cp_1|^2 \leq \left(1 - \frac{\varepsilon + \varepsilon^2}{2}\right)^2 + \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2} + O(\varepsilon^2) < 1,$$

thus D also covers p_1. Finally, we have

$$|cp_3|^2 \leq \left(\frac{\varepsilon + \varepsilon^2}{2}\right)^2 + \left(1 + \varepsilon - \sqrt{\frac{\varepsilon}{2}}\right)^2 \leq 1 - \sqrt{2\varepsilon} + O(\varepsilon) < 1,$$

thus D also covers p_3.

We have shown that $\mathsf{ALG}(\sigma)/\mathsf{OPT}(\sigma) \geq 4$, and the proof is complete. □

3.2 A new lower bound in d-space

We introduce some additional terminology. For every integer k, $0 \leq k < d$, a *k-sphere* of radius r centered at a point $c \in \mathbb{R}^d$ is the locus of points in $\mathbb{R}^d$ at distance r from a center c, and lying in a $(k+1)$-dimensional affine subspace that contains c. In particular, a $(d-1)$-sphere of radius r centered at c is the set of *all* points $p \in \mathbb{R}^d$ such that $|cp| = r$; a 1-sphere is a circle lying in a 2-dimensional affine plane; and a 0-sphere is a pair of points whose midpoint is c. A *k-hemisphere* is a k-dimensional manifold with boundary, defined as the intersection $S \cap H$, where S is a k-sphere centered at some point $c \in \mathbb{R}^d$ and H is a halfspace whose boundary ∂H contains c but does not contain S. For $k \geq 1$, the *relative boundary* of the k-hemisphere $S \cap H$ is the $(k-1)$-sphere $S \cap (\partial H)$ concentric with S; and the *pole* of $S \cap H$ is the unique point $p \in H$ such that $\overrightarrow{cp}$ is orthogonal to the k-dimensional affine subspace that contains $S \cap (\partial H)$. For $k = 0$, a 0-hemisphere consists of a single point, and we define the *pole* to be that point. We make use of the following observation.

Observation 1. *Let S be a k-sphere of radius $1+\varepsilon$, where $0 \le k < d$ and $\varepsilon > 0$; and let B be a unit ball in $\mathbb{R}^d$. Then $S \setminus B$ contains a k-hemisphere.*

Proof. Without loss of generality, S is centered at the origin, and lies in the subspace spanned by the coordinate axes $x_1, \ldots, x_{k+1}$. By symmetry, we may also assume that the center of B is on the nonnegative x_1-axis, say, at $(b, 0, \ldots, 0)$ for some $b \ge 0$. If $b = 0$, then S and B are concentric and B lies in the interior of S, consequently, $S \setminus B = S$. Otherwise, $S \cap B$ lies in the open halfspace $x_1 > 0$, and $S \setminus B$ contains the k-hemisphere $S \cap \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 \le 0\}$. □

Theorem 3. *The competitive ratio of every deterministic online algorithm for* Unit Covering *in $\mathbb{R}^d$ under the L_2 norm is at least $d+1$ for every $d \ge 1$; and at least $d+2$ for $d \in \{2, 3\}$.*

Proof. Consider a deterministic online algorithm ALG. We present an input instance σ for ALG and show that the solution $\mathsf{ALG}(\sigma)$ is at least $d+1$ times $\mathsf{OPT}(\sigma)$. In particular, σ consists of $d+1$ points in $\mathbb{R}^d$ that fit in a unit ball, hence $\mathsf{OPT}(\sigma) = 1$, and we show that ALG is required to place a new unit ball for each point in σ. Similar to the proof of Theorem 2, our proof works like a two-player game between Alice and Bob.

Let the first point $p_0 = o$ be the origin in $\mathbb{R}^d$ (we will use either notation as convenient). For a constant $\varepsilon \in (0, \frac{1}{2d})$, let S_0 be the $(d-1)$-sphere of radius $1+\varepsilon$ centered at the origin o. Refer to Fig. 3. Next, B_0 is placed to cover p_0. The remaining points $p_1, \ldots, p_d$ in σ are chosen adaptively, depending on Bob's moves. We maintain the following two invariants: For $i = 1, \ldots, d$, when Alice has placed points $p_0, \ldots, p_{i-1}$, and Bob placed unit balls $B_0, \ldots B_{i-1}$,

(I) the vectors $\overrightarrow{op_j}$, for $j = 1, \ldots, i-1$, are pairwise orthogonal and they each have length $1+\varepsilon$;

(II) there exists a $(d-i)$-hemisphere $H_i \subset S_0$ that lies in the $(d-i+1)$-dimensional subspace orthogonal to $\langle \overrightarrow{op_j} : j = 1, \ldots, i-1 \rangle$ and is disjoint from $\bigcup_{j=0}^{i-1} B_j$.

Both invariants hold for $i = 1$: (I) is vacuously true, and (II) holds by Observation 1 (the first condition of (II) is vacuous in this case).

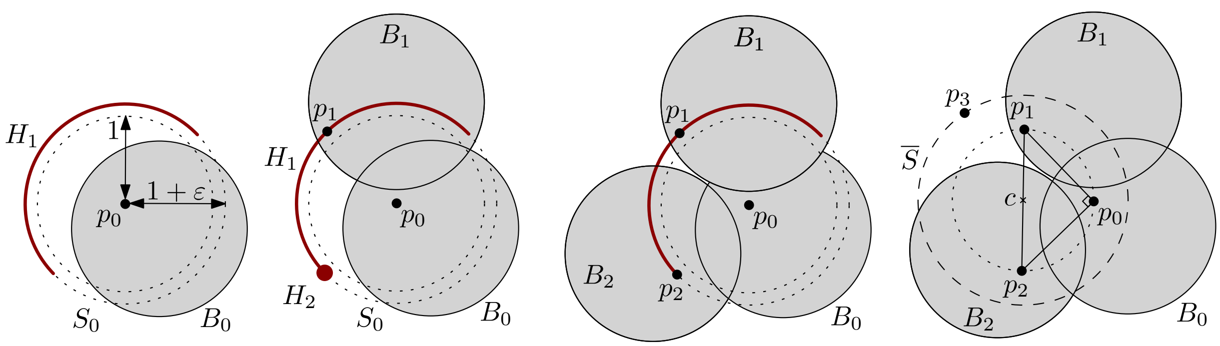

Figure 3: The first three steps of the game between Alice and Bob in the proof of Theorem 3 for $d = 2$. After the 3rd step, Alice can place a 4th point $p_3 \in \overline{S}$ which is not covered by the balls B_0, B_1, B_2.

At the beginning of step i (for $i = 1, \ldots, d$), assume that both invariants hold. Alice chooses p_i to be the pole of the $(d-i)$-hemisphere H_i. By Invariant (II), p_i is not covered by $B_0, \ldots, B_{i-1}$, and Bob has to choose a new unit ball B_i that contains p_i. By Invariant (I), $H_i \subset S_0$, so $|op_i| = 1+\varepsilon$. By Invariant (II), $\overrightarrow{op_i}$ is orthogonal to the vectors $\overrightarrow{op_j}$, for $j = 1, \ldots, i-1$. Hence Invariant (I) is maintained.

Let S_i be the relative boundary of H_i, which is a $(d-i-1)$-sphere centered at the origin. Since p_i is the pole of H_i, $\overrightarrow{op_i}$ is orthogonal to the $(d-i)$-dimensional subspace spanned by S_i.

By Observation 1, S_i contains a $(d-i-1)$-hemisphere that is disjoint from B_i. Denote such a $(d-i-1)$-hemisphere by $H_{i+1} \subset S_i$. Clearly, H_{i+1} is disjoint from the balls $B_0, \ldots, B_{i-1}, B_i$; so Invariant (II) is also maintained.

By construction, p_i $(i = 1, \ldots, d)$ is not covered by the balls $B_0, \ldots, B_{i-1}$, so Bob has to place a unit ball for each of the $d+1$ points $p_0, p_1, \ldots, p_d$. By Invariant (I), the points $p_1, \ldots, p_d$ span a regular $(d-1)$-dimensional simplex of side length $(1+\varepsilon)\sqrt{2}$. Recall that $\varepsilon < 1/(2d)$. By Jung's Theorem [23, p. 46], the radius of the smallest enclosing ball of $p_1, \ldots, p_d$ is

$$R = (1+\varepsilon)\sqrt{2} \cdot \sqrt{\frac{d-1}{2d}} = (1+\varepsilon) \cdot \sqrt{\frac{d-1}{d}} < \left(1 + \frac{1}{2d}\right)\sqrt{\frac{d-1}{d}} = \sqrt{\frac{4d^3 - 3d - 1}{4d^3}} < 1,$$

and this ball contains the origin p_0, as well.

We next show how to adjust the argument to derive a slightly better lower bound of $d+2$ when $d \in \{2, 3\}$. Let B be the smallest enclosing ball of the points $p_0, p_1, \ldots, p_d$, and let c be the center of B. As noted above, the radius of B is $R = (1+\varepsilon)\sqrt{(d-1)/d}$. Let $\overline{S}$ be the $(d-1)$-sphere of radius $2 - R = 2 - (1+\varepsilon)\sqrt{(d-1)/d}$ centered at c. Then the smallest enclosing ball of B and an arbitrary point $p_{d+1} \in \overline{S}$ has unit radius. That is, points $p_0, \ldots, p_d, p_{d+1}$ fit in a unit ball. This raises the question whether Alice can choose yet another point $p_{d+1} \in \overline{S}$ outside of the balls $B_0, \ldots, B_d$ placed by Bob.

For $d = 2$, the radius of $\overline{S}$ is $2 - (1+\varepsilon)\sqrt{1/2} = 2 - (1+\varepsilon)(\sqrt{2}/2) \geq 1.2928$ (provided that $\varepsilon > 0$ is sufficiently small). A unit disk can cover a circular arc in $\overline{S}$ of diameter at most 2. If 3 unit disks can cover $\overline{S}$, then $\overline{S}$ would be the smallest enclosing circle of a triangle of diameter at most 2, and its radius would be at most $\frac{2}{3}\sqrt{3} \leq 1.1548$ by Jung's Theorem. Consequently, Alice can place a 4th point $p_3 \in \overline{S}$ outside of B_0, B_1, B_2, and all four points $p_0, \ldots, p_3$ fit in a unit disk; see Fig. 3 (right) for an example. That is, $\mathsf{ALG}(\sigma) = 4$ and $\mathsf{OPT}(\sigma) = 1$; and we thereby obtain an alternative proof of Theorem 2.

For $d = 3$, the radius of $\overline{S}$ is $R_1 = 2 - (1+\varepsilon)(\sqrt{2/3}) \geq 1.1835$ (provided that $\varepsilon > 0$ is sufficiently small). Let c_i denote the center of B_i, for $i = 0, 1, 2, 3$; we may assume that at least one of the balls B_i, say B_0, is not concentric with $\overline{S}$, since otherwise $\bigcup_{i=0}^{3} B_i$ would cover zero area of $\overline{S}$. We may also assume for concreteness that cc_0 is a vertical segment; let π_0 denote the horizontal plane incident to c. Then $C = \overline{S} \cap \pi_0$ is the horizontal great circle (of radius R_1) of $\overline{C}$, centered at c. Note that $C \cap B_0 = \emptyset$, and so if $\bigcup_{i=0}^{3} B_i$ covers $\overline{S}$, then $\bigcup_{i=1}^{3}(B_i \cap \pi_0)$ covers C. However, the analysis of the planar case ($d = 2$) shows that this is impossible; indeed, we have $R_1 \geq 1.1835 > 1.1548$. Consequently, Alice can place a 5th point $p_4 \in \overline{S}$ outside of $\bigcup_{i=0}^{3} B_i$, and all five points $p_0, \ldots, p_4$ fit in a unit ball. That is, $\mathsf{ALG}(\sigma) = 5$ and $\mathsf{OPT}(\sigma) = 1$ and a lower bound of 5 on the competitive ratio follows. □

3.3 Generalization to centrally symmetric convex bodies

Let C be a d-dimensional convex body in $\mathbb{R}^d$, $d \in \mathbb{N}$. A set of the form $\lambda C + x = \{\lambda c + x \mid c \in C\}$, where $0 < \lambda < 1$ and $x \in \mathbb{R}^d$, is called a *smaller homothetic copy* of C. The *illumination number* of C, denoted $I(C)$, is the minimum number of smaller homothetic copies of C whose union contains C [8, p. 136]. The parameter $I(C)$ is invariant under nondegenerate affine transformations. It was introduced by Levi and Hadwiger, who conjectured that $I(C) \leq 2^d$, with equality if and only if C is a parallelepiped. The conjecture is currently settled in the affirmative for $d = 2$ only. See [2, 3, 8, 30, 31] for the history of the problem, the current best bounds, and equivalent formulations of the illumination number. For instance, it is known that $I(C) \leq 4^d(5d \ln d)$ for any C. If C is centrally symmetric, then $I(C) \leq 2^d(d \ln d + d \ln \ln d + 5d)$; see [8, Ch. 3.3]. It is also known that $I(C) = 2^d$ for any full-dimensional parallelepiped in $\mathbb{R}^d$.

When C is a centrally symmetric convex body in $\mathbb{R}^d$ (i.e., $C = -C$), then C defines a norm, denoted by L_C: The L_C norm of a point $v \in \mathbb{R}^d$ is $\min\{x > 0 : v \in xC\}$. A unit ball under the L_C norm is a translate of C.

Theorem 4. *For every centrally symmetric convex body C in $\mathbb{R}^d$, $d \in \mathbb{N}$, the competitive ratio of every deterministic online algorithm for* Unit Covering *in $\mathbb{R}^d$ under the L_C norm is at least $I(C)$.*

Proof. Let C be a centrally symmetric convex body in $\mathbb{R}^d$, $d \in \mathbb{N}$. We model the online Unit Covering problem as a two-player game between Alice and Bob. Alice constructs a sequence σ of points $p_1, \ldots, p_n \in \mathbb{R}^d$, where $n = I(C)$. After Alice places point p_i, if this point is not covered by previous translates, Bob needs to place a translate of C, denoted C_i, such that $p_i \in C_i$. We claim that Alice can find a point p_i, for $i = 1, \ldots, I(C)$, such that

(I) the union $\bigcup_{j<i} C_j$ does not contain point p_i, and

(II) the set $\{p_1, \ldots, p_i\}$ is contained in the interior of some translate of C.

Condition (II) implies that $P = \{p_1, \ldots, p_{I(C)}\}$ can be covered by one translate of C; and condition (I) implies that Bob is forced to place $I(C)$ translates of C. Consequently, $\mathsf{OPT}(\sigma) = 1$ and $\mathsf{ALG}(\sigma) = I(C)$, hence the competitive ratio of any algorithm that Bob follows is at least $I(C)$.

For $i = 1$, conditions (I) and (II) trivially hold. Assume that both (I) and (II) are satisfied for $i = k$, for some $1 \leq k < I(C)$. It is enough to show that Alice can choose a point p_{k+1} such that (I) and (II) hold for $i = k + 1$.

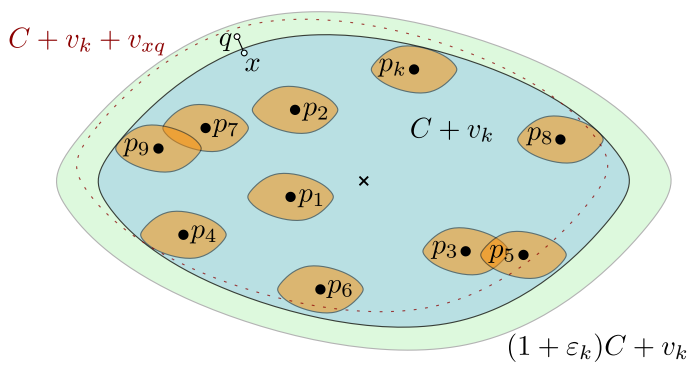

Figure 4: A centrally symmetric convex body $C + v_k$ contains points $p_1, \ldots, p_k$ for $k = 10$. Translates of $\varepsilon_k C$ are centered at each point, which are in turn contained in $(1 + \varepsilon_k)C + v_k$. Points $q \in \text{int}((1 + \varepsilon_k)C + v_k)$ and $x \in \text{int}(C + v_k)$ are at L_C-distance less than ε_k. The translate $C + v_k + v_{xq}$ (in red dotted lines) contains $p_1, \ldots, p_k$, and q.

Let ε_k be the largest value such that translates of $\varepsilon_k\, C$ centered at $p_1, \ldots, p_k$ are contained in some translate of C, that we denote by $C + v_k$, where $v_k \in \mathbb{R}^d$ is the translation vector. We have $\varepsilon_k > 0$ by (II); refer to Fig. 4. Note that $(1 + \varepsilon_k)C + v_k$ is the ε_k-neighborhood of $C + v_k$ in the L_C norm (i.e., the set of all points in $\mathbb{R}^d$ at L_C-distance at most ε_k from $C + v_k$).

We claim that for every point $q \in \text{int}((1+\varepsilon_k)C+v_k)$, the interior of some translate of C contains $\{p_1, \ldots, p_k, q\}$. Indeed, the L_C-distance between q and $\text{int}(C + v_k)$ is less than ε_k. Consequently, there exists a point $x \in \text{int}(C + v_k)$ such that the L_C norm of vector $v_{xq} = q - x$ is less than ε_k. Clearly, $q \in \text{int}(C + v_k + v_{xq})$. By the choice of $\varepsilon_k > 0$, the L_C-distance between the points $p_1, \ldots, p_k$ and $\partial(C + v_k)$ is at least ε_k. This implies that $\text{int}(C + v_k + v_{xq})$ contains the points $p_1, \ldots, p_k$, as well, and completes the proof of the claim.

As the illumination number is an affine invariant, we have $I((1+\varepsilon_k)C) = I(C)$. Since $k < I(C)$, the translates $C_1, \ldots, C_k$ placed by Bob, do not cover $(1+\varepsilon_k)C + v_k$. Consequently, there exists a point $p_{k+1} \in \mathrm{int}((1+\varepsilon_k)C + v_k)$ that is not contained in $\bigcup_{j=1}^{k} C_j$. By the above claim, some translate of C contains both $\{p_1, \ldots, p_k\}$ and p_{k+1} in its interior. □

According to the celebrated Borsuk–Ulam theorem [6] (for instance, see [23, p. 22] or [32]), the d-dimensional Euclidean ball of diameter 1, B_d, cannot be covered by d sets of smaller diameter; in particular, it cannot be covered by d smaller balls, and thus, $I(B_d) \geq d+1$. Consequently, Theorem 4 immediately implies the following result for the L_2 norm in $\mathbb{R}^d$ (obtained earlier via a different proof in Theorem 3).

Corollary 2. *For every $d \in \mathbb{N}$, the competitive ratio of every deterministic online algorithm for* Unit Covering *in $\mathbb{R}^d$ under the L_2 norm is at least $d+1$.*

Recall that $I(C) = 2^d$ when C is a centrally symmetric parallelepiped in $\mathbb{R}^d$. Consequently, Theorem 4 immediately implies the following result for the L_∞ norm in $\mathbb{R}^d$ (obtained in [15] via a different proof).

Corollary 3. *For every $d \in \mathbb{N}$, the competitive ratio of every deterministic online algorithm for* Unit Covering *in $\mathbb{R}^d$ under the L_∞ norm is at least 2^d.*

4 Unit covering of lattice points in two and three dimensions

In this section, we describe optimal deterministic algorithms for the online Unit Covering of points from the infinite unit square and hexagonal lattices in two and three dimensions. We start with the integer lattice $\mathbb{Z}^2$.

Theorem 5. *There exists a deterministic online algorithm for online* Unit Covering *of integer points (points in $\mathbb{Z}^2$) with competitive ratio* 3. *This result is tight: the competitive ratio of any deterministic online algorithm for this problem is at least* 3.

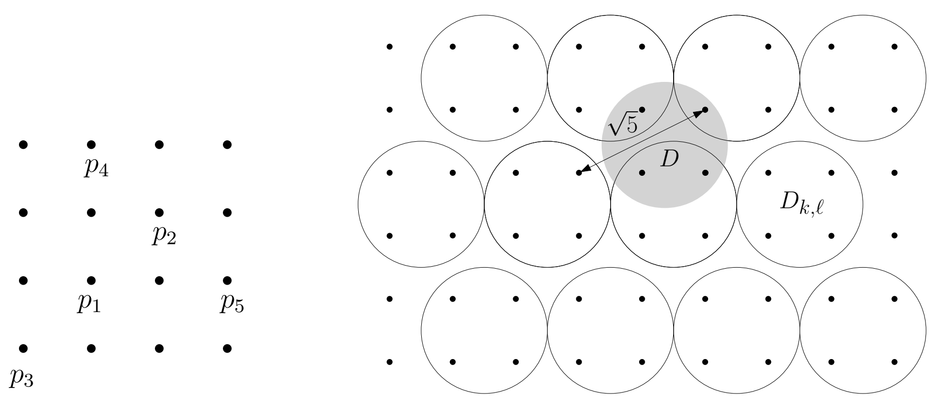

Figure 5: Left: Lower bound for $\mathbb{Z}^2$. Right: Illustration of the upper bound; the disk D is shaded.

Proof. First, we prove the lower bound. A lower bound proof uses the following points in $\mathbb{Z}^2$:

$$p_1 = (0,0), \quad p_2 = (1,1), \quad p_3 = (-1,-1), \quad p_4 = (0,2), \quad p_5 = (2,0);$$

refer to Fig. 5 (left). The adversary first presents point p_1, and the algorithm covers it with some unit disk D_1. Observe that D_1 misses at least one point from $\{p_2, p_3\}$, since $|p_2p_3| = 2\sqrt{2} > 2$. Without loss of generality, we may assume that D_1 missed p_2; and this further implies that D_1 covers neither p_4 nor p_5, otherwise it would also cover p_2, which is a contradiction. Now, the adversary presents p_2, and the algorithm covers it with some unit disk D_2.

If D_2 covers p_4, then the next input point is p_5, otherwise it is p_4. In either case, a third disk is needed. To finish the proof, observe that $\{p_1, p_2, p_4\}$ and $\{p_1, p_2, p_5\}$ can each be covered by a single unit disk; hence the competitive ratio of any deterministic algorithm is at least 3.

Next, we present an algorithm of competitive ratio 3. The integer lattice $\mathbf{\Lambda} := \{s\mathbf{e}_1 + t\mathbf{e}_2 : s, t \in Z\}$ is generated by the standard unit vectors $\mathbf{e}_1, \mathbf{e}_2$. Our algorithm uses disks centered at the points of the lattice $\mathbf{\Xi} := \{s\mathbf{u} + t\mathbf{v} : s, t \in \mathbb{Z}\}$, generated by the vectors $\mathbf{u} = 2\mathbf{e}_1$ and $\mathbf{v} = -\mathbf{e}_1 + 2\mathbf{e}_2$. Partition $\mathbf{\Lambda}$ into 4-element subsets $S_{k,\ell}$, for $k, \ell \in \mathbb{Z}$; refer to Fig. 5 (right). Let $\theta = \pi/4$, and consider the four unit vectors $\mu_0 = (\cos\theta, \sin\theta)$, $\mu_1 = (\cos 3\theta, \sin 3\theta), \mu_2 = -\mu_0, \mu_3 = -\mu_1$. Then,

$$\xi_{k,\ell} := k\mathbf{u} + \ell\mathbf{v}, \text{ for all } k, \ell \in \mathbb{Z},$$
$$S_{k,\ell} := \{\xi_{k,\ell} + \sqrt{0.5}\mu_i \mid i = 0, 1, 2, 3\}.$$

Observe that $\text{conv}(S_{k,\ell})$, the convex hull of $S_{k,\ell}$, is a unit square, which is contained in a concentric unit disk, denoted by $D_{k,\ell}$. The disks $D_{k,\ell}$, for $k, \ell \in \mathbb{Z}$, are pairwise interior-disjoint.

When a point $p \in \mathbb{Z}^2$ arrives, we have $p \in S_{k,\ell}$ for some $k, \ell \in \mathbb{Z}$, and the algorithm covers p with the disk $D_{k,\ell}$. For the analysis, we consider unit disks in an optimal solution. Note that every unit disk D in any optimal solution contains at most four points from $\mathbb{Z}^2$, and if D contains precisely four points from $\mathbb{Z}^2$, then the convex hull of $\text{conv}(D \cap \mathbb{Z}^2)$ is a unit square. A unit disk D cannot intersect four or more sets $S_{k,\ell}$, otherwise it would contain points from two nonadjacent sets $S_{k,\ell}$ and $S_{k',\ell'}$, and the distance between the closest points in such sets is at least $\sqrt{5} > 2$ (see Fig. 5 (right) for an example), contradicting the assumption that D has unit radius. Hence, every disk in an optimal solution contains points from at most three sets of type $S_{k,\ell}$. We conclude that the algorithm has a competitive ratio of 3. □

We now state our result for the infinite hexagonal lattice.

Theorem 6. *There exists a deterministic online algorithm for online* Unit Covering *of points of the hexagonal lattice with competitive ratio* 3*. This result is tight: the competitive ratio of any deterministic online algorithm for this problem is at least* 3*.*

Proof. Let $\theta = \pi/3$, and consider the six unit vectors $\mu_i = (\cos i\theta, \sin i\theta)$, for $i = 0, 1, 2, 3, 4, 5$; note that $\mu_i = -\mu_{i+3}$ for every i, where the indexes are modulo 6. Put $\mathbf{u} = \mu_0$, and $\mathbf{v} = \mu_1$. The hexagonal lattice $\mathbf{\Lambda} = \{s\mathbf{u} + t\mathbf{v} : s, t \in \mathbb{Z}\}$ is generated by $\mathbf{u}$ and $\mathbf{v}$.

We first prove the lower bound of 3. A lower bound proof uses the following points; refer to Fig. 6 (left).

$$p_1 = \mathbf{0}, \quad p_2 = -\mathbf{u} - \mathbf{v}, \quad p_3 = \mathbf{u} + \mathbf{v}, \quad p_4 = 2\mathbf{u}, \quad p_5 = -\mathbf{u} + 2\mathbf{v}, \quad p_6 = 2\mathbf{v}, \quad p_7 = 2\mathbf{u} - \mathbf{v}.$$

The first point, p_1, arrives and D_1 is placed to cover it. D_1 misses at least one of $\{p_2, p_3\}$, since $|p_2p_3| = |2(\mathbf{u} + \mathbf{v})| = 2\sqrt{3} > 2$. By symmetry, we may assume that D_1 misses p_3. Now, point p_3 arrives and D_2 is placed to cover it. We distinguish two cases:

Case 1: D_2 misses p_4. Since D_1 misses p_3, D_1 also misses p_4. Otherwise, if D_1 covers p_4, then D_1 also covers p_3, a contradiction. The algorithm uses D_3 to cover p_4. Thus the ratio is 3 since $\{p_1, p_3, p_4\}$ can be covered by a single disk centered at $(p_1 + p_4)/2$, and the algorithm has used three disks: D_1, D_2, and D_3.

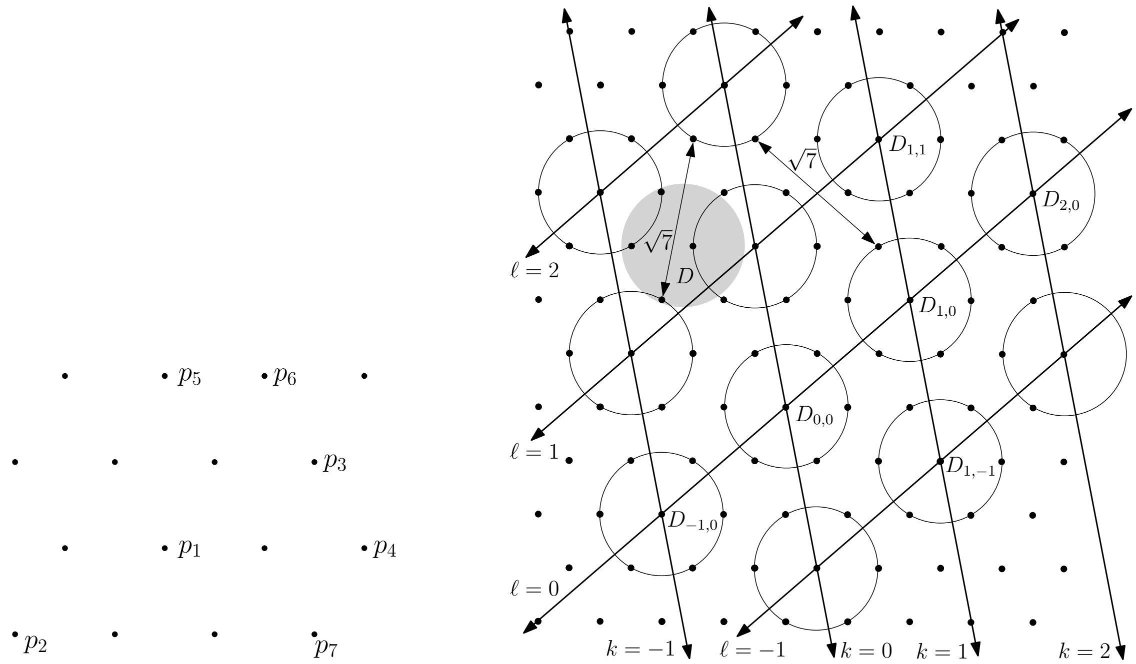

Figure 6: Left: Lower bound for the hexagonal lattice. Right: Illustration of the upper bound.

Case 2: D_2 covers p_4. This means that D_2 misses p_5. Two situations may occur:

Case 2(a): D_1 misses p_5 too. Then p_5 is the next input point, and the algorithm uses D_3 to cover it. Here $\{p_1, p_3, p_5\}$ can be covered by a single disk centered at $(p_1 + p_3 + p_5)/3$, but the algorithm has used three disks: D_1, D_2, and D_3.

Case 2(b): D_1 covers p_5. Since D_1 does not cover p_3, D_1 cannot cover p_6. If D_2 misses p_6, let p_6 be the third point presented; the algorithm uses D_3 to cover p_6. Here $\{p_1, p_3, p_6\}$ can be covered by a single disk centered at $(p_1 + p_6)/2$, but the algorithm has used three disks: D_1, D_2, and D_3. If D_2 covers p_6, let p_7 be the third point presented. Note that D_1 cannot cover p_7 since it covers p_5; also, D_2 cannot cover p_7 since it covers p_6. The algorithm uses D_3 to cover p_7. Here $\{p_1, p_3, p_7\}$ can be covered by a single disk centered at $(p_1 + p_3 + p_7)/3$, but the algorithm has used three disks: D_1, D_2, and D_3.

In all cases a lower bound of 3 has been enforced by Alice, as required.

Next, we present an algorithm having competitive ratio of 3. Our algorithm uses disks centered at the sublattice $\mathbf{\Lambda}' = \{s\mathbf{u}' + t\mathbf{v}' : s, t \in \mathbb{Z}\}$, generated by the vectors $\mathbf{u}' = \mathbf{u} + 2\mathbf{v}$ and $\mathbf{v}' = -2\mathbf{u} + 3\mathbf{v}$ of length $|\mathbf{u}'| = |\mathbf{v}'| = \sqrt{7}$; see Fig. 6 (right) for an illustration. We partition the lattice points into 7-element subsets $S_{k,\ell}$, for $k, \ell \in \mathbb{Z}$; refer to Fig. 6 (right). Let

$$\xi_{k,\ell} := k\mathbf{u}' + \ell\mathbf{v}' = (k - 2\ell)\,\mathbf{u} + (2k + 3\ell)\,\mathbf{v}, \text{ for all } k, \ell \in \mathbb{Z},$$
$$S_{k,\ell} := \{\xi_{k,\ell}\} \cup \{\xi_{k,\ell} + \mu_i \mid i = 0, 1, 2, 3, 4, 5\}.$$

The seven points of $S_{k,\ell}$ form a regular hexagon and its center $\xi_{k,\ell}$; $S_{k,\ell}$ is contained in a unique unit disk denoted by $D_{k,\ell}$. These disks are pairwise disjoint and cover all lattice points. Two sets $S_{k,\ell}$ are *adjacent* if their centers are at distance $|\mathbf{u} + 2\mathbf{v}| = \sqrt{7}$; in particular, each set $S_{k,\ell}$ is adjacent to six other sets. Importantly, the distance between any two nonadjacent sets exceeds 2. Indeed, the distance between the centers of any two nonadjacent sets, $S_{k,\ell}$ and $S_{k',\ell'}$, is at least

$|\mathbf{u}+2\mathbf{v}-2\mathbf{u}+3\mathbf{v}| = |-\mathbf{u}+5\mathbf{v}| = \sqrt{7\cdot 3} = \sqrt{21}$ (for example, the distance between the centers of $S_{0,0}$ and $S_{2,-1}$). All points in $S_{k,\ell}$ and $S_{k',\ell'}$, respectively, are within unit distance from their centers. By the triangle inequality, the distance between any two points in $S_{k,\ell}$ and $S_{k',\ell'}$ is at least $\sqrt{21}-2>2$.

When a lattice point p arrives, we have $p \in S_{k,\ell}$ for some $k,\ell \in \mathbb{Z}$, and the algorithm covers p with the disk $D_{k,\ell}$. For the analysis, consider a unit disk D from an optimal solution. If D intersects two sets, $S_{k,\ell}$ and $S_{k',\ell'}$, then they are adjacent. Consequently, the sets intersecting D are pairwise adjacent, hence D intersects at most three sets of type $S_{k,\ell}$. This proves that our algorithm has a competitive ratio of 3. □

The partitioning method used for covering points in $\mathbb{Z}^2$ can be extended to $\mathbb{Z}^3$; however, in this case we do not obtain a tight bound. (A lower bound of 3 follows from Theorem 5.)

Theorem 7. *There exists a deterministic online algorithm for online* Unit Covering *of points in* $\mathbb{Z}^3$ *with competitive ratio* 5.

Proof. The cubic integer lattice $\mathbf{\Lambda} := \{q\mathbf{e}_1+s\mathbf{e}_2+t\mathbf{e}_3 : q,s,t \in \mathbb{Z}\}$ is generated by the standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Our algorithm uses balls centered at the points of the lattice $\mathbf{\Xi} := \{q\mathbf{u}+s\mathbf{v}+t\mathbf{w} : q,s,t \in \mathbb{Z}\}$, generated by the vectors $\mathbf{u}=2\mathbf{e}_1$, $\mathbf{v}=\mathbf{e}_1+2\mathbf{e}_2$, and $\mathbf{w}=\mathbf{e}_2+2\mathbf{e}_3$. Partition $\mathbf{\Lambda}$ into 8-element subsets $S_{k,\ell,m}$, for $k,\ell,m \in \mathbb{Z}$. Let $\theta=\pi/4$, and consider the four unit vectors $\mu_0=(\cos\theta,\sin\theta)$, $\mu_1=(\cos 3\theta,\sin 3\theta)$, $\mu_2=-\mu_0$, $\mu_3=-\mu_1$. Then,

$$\xi_{k,\ell,m} := k\mathbf{u}+\ell\mathbf{v}+m\mathbf{w}, \text{ for all } k,\ell,m \in \mathbb{Z},$$
$$S_{k,\ell,m} := \{\xi_{k,\ell,m}+\sqrt{0.5}\mu_i+0.5\mathbf{e}_3 \mid i=0,1,2,3\} \cup \{\xi_{k,\ell,m}+\sqrt{0.5}\mu_i-0.5\mathbf{e}_3 \mid i=0,1,2,3\}.$$

The convex hull of $S_{k,\ell,m}$ is a unit cube, which is contained in a concentric unit ball, denoted $B_{k,\ell,m}$. The balls $B_{k,\ell,m}$, for $k,\ell,m \in \mathbb{Z}$, are pairwise interior-disjoint. The infinite collection of unit balls $B_{k,\ell,m}$, for some fixed m form a *layer*. The balls in a layer are arranged following the pattern illustrated in Fig. 5 (right); the figure shows a horizontal cross-section. An orthogonal projection of the balls appears in Fig. 7. The gray and yellow layers are placed alternatively. Every point in $\mathbb{Z}^3$ is covered by exactly one ball from a specific layer (gray/yellow).

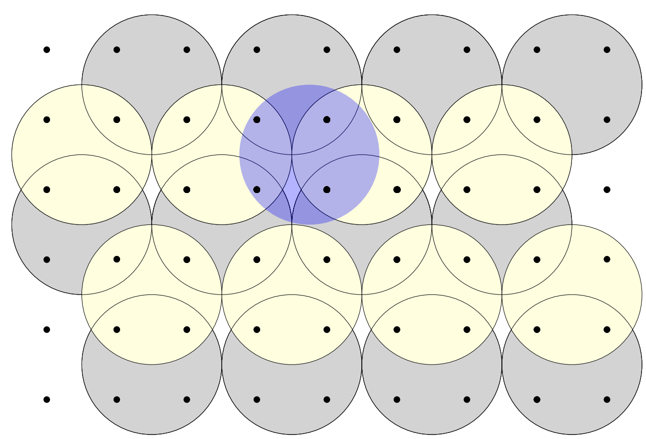

Figure 7: Illustration of the upper bound for covering points in $\mathbb{Z}^3$. The gray balls belong to one layer, and the yellow ones to another. The blue shaded ball covers points from three gray balls and two yellow balls.

When a point $p \in \mathbb{Z}^3$ arrives, we have $p \in S_{k,\ell,m}$ for some $k,\ell,m \in \mathbb{Z}$, and we cover p with the ball $B_{k,\ell,m}$. For the analysis, we consider unit balls in an optimal solution. Note that every unit ball B in an optimal solution contains at most eight points from $\mathbb{Z}^3$. If these eight points belong to the same layer, then the ratio is at most 3; this follows directly from Theorem 5. Otherwise, the points must belong to two consecutive layers, in which case, the unit ball B intersects at most three sets of type $S_{k,\ell,m}$ in any layer, and if it intersects three such sets in one layer, it can intersect at most two such sets from the other layer, due to misalignment. Since every ball in an optimal solution contains points from at most five sets of type $S_{k,\ell,m}$, the algorithm has a competitive ratio of 5. □

5 Conclusion

Our results suggest several directions for future study. We summarize a few specific questions of interest. By Theorem 1 and a remark in the Introduction, `Algorithm Centered` has a competitive ratio $O(1.321^d)$ also for UNIT CLUSTERING in $\mathbb{R}^d$ under the L_2 norm. However, presently there is no superlinear lower bound.

Problem 4. *Is there a lower bound on the competitive ratio for* UNIT CLUSTERING *in* $\mathbb{R}^d$ *under the* L_2 *norm that is exponential in* d*? Is there a superlinear lower bound?*

Problem 5. *Is there a lower bound on the competitive ratio for* UNIT COVERING *in* $\mathbb{R}^d$ *under the* L_2 *norm that is exponential in* d*? Is there a superlinear lower bound?*

Problem 6. *Can the online algorithm for* UNIT COVERING *of integer points be extended to higher dimensions? What ratio can be obtained for this variant for large* d*?*

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