Piercing translates and homothets of a convex body*

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Abstract

According to a classical result of Grünbaum, the transversal number $\tau(\mathcal{F})$ of any family $\mathcal{F}$ of pairwise-intersecting translates or homothets of a convex body C in $\mathbb{R}^d$ is bounded by a function of d. Denote by $\alpha(C)$ (resp. $\beta(C)$) the supremum of the ratio of the transversal number $\tau(\mathcal{F})$ to the packing number $\nu(\mathcal{F})$ over all finite families $\mathcal{F}$ of translates (resp. homothets) of a convex body C in $\mathbb{R}^d$. Kim et al. recently showed that $\alpha(C)$ is bounded by a function of d for any convex body C in $\mathbb{R}^d$, and gave the first bounds on $\alpha(C)$ for convex bodies C in $\mathbb{R}^d$ and on $\beta(C)$ for convex bodies C in the plane.

Here we show that $\beta(C)$ is also bounded by a function of d for any convex body C in $\mathbb{R}^d$, and present new or improved bounds on both $\alpha(C)$ and $\beta(C)$ for various convex bodies C in $\mathbb{R}^d$ for all dimensions d. Our techniques explore interesting inequalities linking the covering and packing densities of a convex body. Our methods for obtaining upper bounds are constructive and lead to efficient constant-factor approximation algorithms for finding a minimum-cardinality point set that pierces a set of translates or homothets of a convex body.

Keywords: Geometric transversals, Gallai-type problems, packing and covering, approximation algorithms.

1 Introduction

A *convex body* is a compact convex set in $\mathbb{R}^d$ with nonempty interior. Let $\mathcal{F}$ be a family of convex bodies. The *packing number* $\nu(\mathcal{F})$ is the maximum cardinality of a set of pairwise-disjoint convex bodies in $\mathcal{F}$, and the *transversal number* $\tau(\mathcal{F})$ is the minimum cardinality of a set of points that intersects every convex body in $\mathcal{F}$.

Let G be the *intersection graph* of $\mathcal{F}$ with one vertex for each convex body in $\mathcal{F}$ and with an edge between two vertices if and only if the two corresponding convex bodies intersect. The *independence number* $\alpha(G)$ is the maximum cardinality of an independent set in G. The *clique partition number* $\vartheta(G)$ is the minimum number of classes in a partition of the vertices of G into cliques. Since a set of pairwise-disjoint convex bodies in $\mathcal{F}$ corresponds to an independent set in G, we have $\nu(\mathcal{F}) = \alpha(G)$. Also, since any subset of convex bodies in $\mathcal{F}$ that share a common point corresponds to a clique in G, we have $\tau(\mathcal{F}) \geq \vartheta(G)$. For the special case that $\mathcal{F}$ is a family of axis-parallel boxes in $\mathbb{R}^d$, we indeed have $\tau(\mathcal{F}) = \vartheta(G)$ since any subset of pairwise-intersecting boxes share a common point. In general, we clearly have the inequality $\vartheta(G) \geq \alpha(G)$, thus also $\tau(\mathcal{F}) \geq \nu(\mathcal{F})$. But what else can be said about the relation between $\tau(\mathcal{F})$ and $\nu(\mathcal{F})$?

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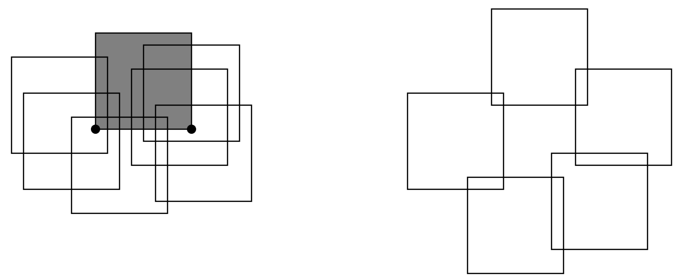

Figure 1: Piercing a family $\mathcal{F}$ of axis-parallel unit squares. Left: all squares that intersect the highest (shaded) square contain one of its two lower vertices. Right: five squares form a 5-cycle.

For example, let $\mathcal{F}$ be any finite family of axis-parallel unit squares in the plane, and refer to Figure 1. One can obtain a subset of pairwise-disjoint squares by repeatedly selecting the highest square that does not intersect the previously selected squares. Then $\mathcal{F}$ is pierced by the set of points consisting of the two lower vertices of each square in the subset. This implies that $\tau(\mathcal{F}) \le 2 \cdot \nu(\mathcal{F})$. The factor of 2 cannot be improved below $\frac{3}{2}$ since $\tau(\mathcal{F}) = 3$ and $\nu(\mathcal{F}) = 2$ for a family $\mathcal{F}$ of five squares arranged into a 5-cycle [14].

For a convex body C in $\mathbb{R}^d$, $d \ge 2$, define

$$\alpha(C) := \sup_{\mathcal{F}_t} \frac{\tau(\mathcal{F}_t)}{\nu(\mathcal{F}_t)} \quad \text{and} \quad \beta(C) := \sup_{\mathcal{F}_h} \frac{\tau(\mathcal{F}_h)}{\nu(\mathcal{F}_h)},$$

where $\mathcal{F}_t$ ranges over all finite families of translates of C, and $\mathcal{F}_h$ ranges over all finite families of (positive) homothets of C. Our previous discussion (Figure 1) yields the bounds $\frac{3}{2} \le \alpha(C) \le 2$ for any square C.

Define $\alpha_1(C)$ (resp. $\beta_1(C)$) as the smallest number k such that for any finite family $\mathcal{F}$ of *pairwise-intersecting* translates (resp. homothets) of a convex body C, there exists a set of k points that intersects every member of $\mathcal{F}$. Note that α and β generalize α_1 and β_1. For any convex body C, the four numbers $\alpha(C)$, $\beta(C)$, $\alpha_1(C)$, and $\beta_1(C)$ are invariant under any non-singular affine transformation of C, and we have the four inequalities $\alpha_1(C) \le \alpha(C)$, $\beta_1(C) \le \beta(C)$, $\alpha_1(C) \le \beta_1(C)$, and $\alpha(C) \le \beta(C)$.

A classical result by Grünbaum [13] implies that for any convex body C in $\mathbb{R}^d$, both $\alpha_1(C)$ and $\beta_1(C)$ are bounded by functions of d. Deriving bounds on $\alpha_1(C)$ and $\beta_1(C)$ for various types of convex bodies C in $\mathbb{R}^d$ is typical of classic Gallai-type problems [10, 32], and has been extensively studied. For example, Karasev [17] showed that for any family of pairwise-intersecting translates of a convex body in the plane, there always exists a set of three points that intersects every member of the family. This implies that $\alpha_1(C) \le 3$ for any convex body C in the plane. It is folklore that $\alpha_1(C) = \beta_1(C) = 1$ for any parallelogram C; see [13] and the references therein. Also, $\alpha_1(C) = 2$ for any affinely regular hexagon C [13, Example 2], $\alpha_1(C) \le \beta_1(C) \le 3$ for any triangle C [7], $3 = \alpha_1(C) < \beta_1(C) = 4$ for any (circular) disk C [13, 9], and $3 = \alpha_1(C) \le \beta_1(C) \le 7$ for any centrally symmetric convex body C in the plane [13]. Perhaps the most celebrated result on point transversals of convex sets is Alon and Kleitman's solution to the Hadwiger-Debrunner (p, q)-problem [2]. We refer to the two surveys [10, pp. 142–150] and [32, pp. 77–78] for more related results.

The two numbers $\alpha_1(C)$ and $\beta_1(C)$ bound the values of $\tau(\mathcal{F})$ for special families $\mathcal{F}$ of translates and homothets of a convex body C with $\nu(\mathcal{F}) = 1$. It is thus natural to study the general case $\nu(\mathcal{F}) \ge 1$, and to obtain estimates on $\alpha(C)$ and $\beta(C)$. Despite the many previous bounds on $\alpha_1(C)$ and $\beta_1(C)$ [10, 32], first estimates on $\alpha(C)$ and $\beta(C)$ have been only obtained recently [20]. Note that the related problem for families of d-intervals (which are nonconvex) has been extensively studied [31, 19, 16, 1, 23].

Kim et al. [20] showed that $\alpha(C)$ is bounded by a function of d for any convex body C in $\mathbb{R}^d$, and gave the first bounds on $\alpha(C)$ for convex bodies C in $\mathbb{R}^d$ and on $\beta(C)$ for convex bodies C in the plane. In this paper, we show that $\beta(C)$ is also bounded by a function of d for any convex body C in $\mathbb{R}^d$, and present new or improved bounds on both $\alpha(C)$ and $\beta(C)$ for various convex bodies C in $\mathbb{R}^d$ for all dimensions d.

Note that in the definitions of α and β, both the convexity of C and the homothety of $\mathcal{F}_t$ and $\mathcal{F}_h$ are necessary for the values $\alpha(C)$ and $\beta(C)$ to be bounded. To see the necessity of convexity, let C be the union of a vertical line segment with endpoints $(0,0)$ and $(0,1)$ and a horizontal line segment with endpoints $(0,0)$ and $(1,0)$, where the shared endpoint $(0,0)$ is the *corner*, and let $\mathcal{F}$ be a family of n translates of C with corners at $(i/n, -i/n)$, $1 \le i \le n$ [14]. Then at least $\lceil n/2 \rceil$ points are required to intersect every member of $\mathcal{F}$. To see the necessity of homothety, let $\mathcal{F}$ be a family of n pairwise intersecting line segments (or very thin rectangles) in the plane such that no three have a common point. Then again at least $\lceil n/2 \rceil$ points are required to intersect every member of $\mathcal{F}$.

Definitions. For a convex body C in $\mathbb{R}^d$, denote by $|C|$ the Lebesgue measure of C, i.e., the area in the plane, or the volume in d-space for $d \ge 3$. For a family $\mathcal{F}$ of convex bodies in $\mathbb{R}^d$, denote by $|\mathcal{F}|$ the Lebesgue measure of the union of the convex bodies in $\mathcal{F}$, i.e., $|\bigcup_{C \in \mathcal{F}} C|$.

For two convex bodies A and B in $\mathbb{R}^d$, denote by $\kappa(A, B)$ the smallest number κ such that A can be covered by κ translates of B, and denote by $A + B = \{a + b \mid a \in A, b \in B\}$ the Minkowski sum of A and B. For a convex body C in $\mathbb{R}^d$, denote by $\lambda C = \{\lambda c \mid c \in C\}$ the *scaled copy* of C by a factor of $\lambda \in \mathbb{R}$, denote by $-C = \{-c \mid c \in C\}$ the *reflexion* of C about the origin, and denote by $C + p = \{c + p \mid c \in C\}$ the *translate* of C by the vector from the origin to a point p (we call p the *reference point* of $C + p$). Write $C - C$ for $C + (-C)$.

For two parallelepipeds P and Q in $\mathbb{R}^d$ that are parallel to each other (but are not necessarily axis-parallel or orthogonal), denote by $\lambda_i(P, Q)$, $1 \le i \le d$, the length ratios of the edges of Q to the corresponding parallel edges of P. Then, for a convex body C in $\mathbb{R}^d$, define

$$\gamma(C) := \min_{P,Q} \left(\lceil \lambda_d(P,Q) \rceil \prod_{i=1}^{d-1} \lceil \lambda_i(P,Q) + 1 \rceil \right), \tag{1}$$

where P and Q range over all pairs of parallelepipeds in $\mathbb{R}^d$ that are parallel to each other, such that $P \subseteq C \subseteq Q$. Note that in this case $\lambda_i(P,Q) \ge 1$ for $1 \le i \le d$.

We next review some definitions concerning packing and covering densities. A family $\mathcal{F}$ of convex bodies is a *packing* in a domain $Y \subseteq \mathbb{R}^d$ if $\bigcup_{C \in \mathcal{F}} C \subseteq Y$ and the convex bodies in $\mathcal{F}$ are pairwise-disjoint[1]; $\mathcal{F}$ is a *covering* of Y if $Y \subseteq \bigcup_{C \in \mathcal{F}} C$. The *density* of a family $\mathcal{F}$ relative to a bounded domain Y is

$$\rho(\mathcal{F}, Y) := \frac{\sum_{C \in \mathcal{F}} |C \cap Y|}{|Y|}.$$

If $Y = \mathbb{R}^d$ is the whole space, then the *upper density* and the *lower density* of $\mathcal{F}$ are, respectively,

$$\overline{\rho}(\mathcal{F}, \mathbb{R}^d) := \limsup_{r \to \infty} \rho(\mathcal{F}, B^d(r)) \quad \text{and} \quad \underline{\rho}(\mathcal{F}, \mathbb{R}^d) := \liminf_{r \to \infty} \rho(\mathcal{F}, B^d(r)),$$

where $B^d(r)$ denote a ball of radius r centered at the origin (since we are taking the limit as $r \to \infty$, a hypercube of side length r can be used instead of a ball of radius r). For a convex body C in $\mathbb{R}^d$, define the *packing density* of C as

$$\delta(C) := \sup_{\mathcal{F} \text{ packing}} \overline{\rho}(\mathcal{F}, \mathbb{R}^d),$$

[1] Our definition of a *packing* is slightly different from the standard definition. In the standard definition, two members of a packing must be interior-disjoint but can still share boundary points; see [26, Definition 3.1] and [5, Section 1.1]. Here we choose the stronger constraint that no two members of a packing can have any point in common, so that the definition of a packing is consistent with the definition of the packing number ν. This small difference does not affect the definitions of packing densities because the latter use limits and suprema.

where $\mathcal{F}$ ranges over all packings in $\mathbb{R}^d$ with congruent copies of C, and define the *covering density* of C as

$$\theta(C) := \inf_{\mathcal{F} \text{ covering}} \underline{\rho}(\mathcal{F}, \mathbb{R}^d),$$

where $\mathcal{F}$ ranges over all coverings of $\mathbb{R}^d$ with congruent copies of C. If the members of $\mathcal{F}$ are restricted to translates of C, then we have the *translative* packing and covering densities $\delta_T(C)$ and $\theta_T(C)$. If the members of $\mathcal{F}$ are further restricted to translates of C by vectors of a lattice, then we have the *lattice* packing and covering densities $\delta_L(C)$ and $\theta_L(C)$. Note that the four densities $\theta_T(C)$, $\theta_L(C)$, $\delta_T(C)$, and $\delta_L(C)$ are invariant under any non-singular affine transformation of C. For any convex body C in $\mathbb{R}^d$, we have the inequalities $\delta_L(C) \le \delta_T(C) \le \delta(C) \le 1 \le \theta(C) \le \theta_T(C) \le \theta_L(C)$.

Main results. Kim et al. [20] recently proved that, for any finite family $\mathcal{F}$ of translates of a convex body in $\mathbb{R}^d$, $\tau(\mathcal{F}) \le 2^{d-1} d^d \cdot \nu(\mathcal{F})$, in particular $\tau(\mathcal{F}) \le 108 \cdot \nu(\mathcal{F})$ when $d = 3$, and moreover $\tau(\mathcal{F}) \le 8 \cdot \nu(\mathcal{F}) - 5$ when $d = 2$. We improve these bounds for all dimensions d in the following theorem. Recall the definition of $\gamma(C)$ in (1).

Theorem 1. *For any finite family $\mathcal{F}$ of translates of a convex body C in $\mathbb{R}^d$,*

$$\tau(\mathcal{F}) \le \gamma(C) \cdot \nu(\mathcal{F}), \quad \text{where } \gamma(C) \le d(d+1)^{d-1}. \tag{2}$$

In particular, $\tau(\mathcal{F}) \le 48 \cdot \nu(\mathcal{F})$ when $d = 3$, and $\tau(\mathcal{F}) \le 6 \cdot \nu(\mathcal{F})$ when $d = 2$.

For any parallelepiped C in $\mathbb{R}^d$, we can take two parallelepipeds P and Q such that $P = C = Q$. Then $\lambda_i(P, Q) = 1$ for $1 \le i \le d$, and $\gamma(C) = 2^{d-1}$. This implies the following corollary:

Corollary 1. *For any finite family $\mathcal{F}$ of translates of a parallelepiped in $\mathbb{R}^d$, $\tau(\mathcal{F}) \le 2^{d-1} \cdot \nu(\mathcal{F})$.*

In contrast, for any finite family $\mathcal{F}$ of (not necessarily congruent or similar) axis-parallel parallelepipeds in $\mathbb{R}^d$, the current best upper bound [11] (see also [18, 19, 25]) is

$$\tau(\mathcal{F}) \le \nu(\mathcal{F}) \log^{d-2} \nu(\mathcal{F}) (\log \nu(\mathcal{F}) - 1/2) + d.$$

Kim et al. [20] also proved that, for any finite family $\mathcal{F}$ of translates of a centrally symmetric convex body in the plane, $\tau(\mathcal{F}) \le 6 \cdot \nu(\mathcal{F}) - 3$. The following theorem gives a general bound for any centrally symmetric convex body in $\mathbb{R}^d$ and an improved bound (if $\nu(\mathcal{F}) \ge 5$) for any centrally symmetric convex body in the plane:

Theorem 2. *For any finite family $\mathcal{F}$ of translates of a centrally symmetric convex body S in $\mathbb{R}^d$,*

$$\tau(\mathcal{F}) \le 2^d \cdot \frac{\theta_L(S)}{\delta_L(S)} \cdot \nu(\mathcal{F}). \tag{3}$$

Moreover, $\tau(\mathcal{F}) \le 24 \cdot \nu(\mathcal{F})$ when $d = 3$, and $\tau(\mathcal{F}) \le \frac{16}{3} \cdot \nu(\mathcal{F})$ when $d = 2$.

For special types of convex bodies in the plane, the following theorem gives sharper bounds than the bounds implied by Theorem 1 and Theorem 2. Also, as we will show later, inequality (4) below may give a better asymptotic bound than (2) and (3) in high dimensions.

Theorem 3. *Let $\mathcal{F}$ be a finite family of translates of a convex body C in $\mathbb{R}^d$. Then*

$$\tau(\mathcal{F}) \le \min_L \kappa((C - C) \cap L, C) \cdot \nu(\mathcal{F}), \tag{4}$$

where L ranges over all closed half spaces bounded by hyperplanes through the center of $C - C$. Moreover, $\tau(\mathcal{F}) \le 4 \cdot \nu(\mathcal{F}) - 1$ if C is a centrally symmetric convex body in the plane. Also,

(i) *If C is a square, then $\tau(\mathcal{F}) \le 2 \cdot \nu(\mathcal{F}) - 1$,*

(ii) *If C is a triangle, then $\tau(\mathcal{F}) \le 5 \cdot \nu(\mathcal{F}) - 2$,*

(iii) *If C is a disk, then $\tau(\mathcal{F}) \le 4 \cdot \nu(\mathcal{F}) - 1$.*

Having presented our bounds for families of translates, we now turn to families of homothets. Kim et al. [20] proved that, for any finite family $\mathcal{F}$ of homothets of a convex body C in the plane, $\tau(\mathcal{F}) \le 16 \cdot \nu(\mathcal{F})$ and, if C is centrally symmetric, $\tau(\mathcal{F}) \le 9 \cdot \nu(\mathcal{F})$. The following theorem gives a general bound for any convex body in $\mathbb{R}^d$, an improved bound for any centrally symmetric convex body in the plane, and additional bounds for special types of convex bodies in the plane:

Theorem 4. *Let $\mathcal{F}$ be a finite family of homothets of a convex body C in $\mathbb{R}^d$. Then*

$$\tau(\mathcal{F}) \le \kappa(C - C, C) \cdot \nu(\mathcal{F}). \tag{5}$$

In particular, $\tau(\mathcal{F}) \le 7 \cdot \nu(\mathcal{F})$ if C is a centrally symmetric convex body in the plane. Moreover,

(i) *If C is a square, then $\tau(\mathcal{F}) \le 4 \cdot \nu(\mathcal{F}) - 3$,*

(ii) *If C is a triangle, then $\tau(\mathcal{F}) \le 12 \cdot \nu(\mathcal{F}) - 9$,*

(iii) *If C is a disk, then $\tau(\mathcal{F}) \le 7 \cdot \nu(\mathcal{F}) - 3$.*

For any parallelepiped C in $\mathbb{R}^d$, $C - C$ is a translate of $2C$, which is the union of 2^d translates of C. Thus $\kappa(C - C, C) = 2^d$. This implies the following corollary:

Corollary 2. *For any finite family $\mathcal{F}$ of homothets of a parallelepiped in $\mathbb{R}^d$, $\tau(\mathcal{F}) \le 2^d \cdot \nu(\mathcal{F})$.*

Both Theorem 3 and Theorem 4 are obtained by a simple greedy method, used also previously by Kim et al. [20]. Although we have improved their bounds using new techniques in Theorem 1 and Theorem 2, we show that a refined analysis of the simple greedy method yields even better asymptotic bounds for high dimensions in Theorem 3 and Theorem 4. We will use the following lemma by Chakerian and Stein [7] in our analysis:

Lemma 1 (Chakerian and Stein [7]). *For every convex body C in $\mathbb{R}^d$ there exist two parallelepipeds P and Q such that $P \subseteq C \subseteq Q$, where P and Q are homothetic with ratio at most d.*

For any convex body C in $\mathbb{R}^d$, let P and Q be the two parallelepipeds in Lemma 1. Since $C - C \subseteq Q - Q$ and $P \subseteq C$, it follows that $\kappa(C - C, C) \le \kappa(Q - Q, P) = \kappa(2Q, P) \le (2d)^d$; see also [20, Lemma 4]. The classic survey by Danzer, Grünbaum, and Klee [10, pp. 146–147] lists several other upper bounds due to Rogers and Danzer: (i) $\kappa(C - C, C) \le \frac{2^d}{d+1} 3^{d+1} \theta_T(C)$ for any convex body C in $\mathbb{R}^d$, (ii) $\kappa(C - C, C) \le 5^d$ and $\kappa(C - C, C) \le 3^d \theta_T(C)$ for any centrally symmetric convex body C in $\mathbb{R}^d$. Note that $\theta_T(C) < d \ln d + d \ln \ln d + 5d = O(d \log d)$ for any convex body C in $\mathbb{R}^d$, according to a result of Rogers [28]. The following lemma summarizes the upper bounds on $\kappa(C - C, C)$:

Lemma 2. *For any convex body C in $\mathbb{R}^d$, $\kappa(C - C, C) \le \min\{(2d)^d, \frac{2^d}{d+1} 3^{d+1} \theta_T(C)\} = O(6^d \log d)$. Moreover, if C is centrally symmetric, then $\kappa(C - C, C) \le \min\{5^d, 3^d \theta_T(C)\} = O(3^d d \log d)$.*

From Lemma 2 and Theorem 4, it follows that $\beta(C)$ is bounded by a function of d, namely by $O(6^d \log d)$, for any convex body C in $\mathbb{R}^d$. Since $\min_L \kappa((C - C) \cap L, C) \le \kappa(C - C, C)$, Lemma 2 also provides upper bounds on $\min_L \kappa((C - C) \cap L, C)$ in Theorem 3. As a result, (4) implies an upper bound $\tau(\mathcal{F}) \le O(6^d \log d) \cdot \nu(\mathcal{F})$ for any finite family $\mathcal{F}$ of translates of a convex body in $\mathbb{R}^d$, which is better than the upper bound $\tau(\mathcal{F}) \le d(d+1)^{d-1} \cdot \nu(\mathcal{F})$ in (2) when d is sufficiently large. Also, (4) implies an upper bound $\tau(\mathcal{F}) \le 3^d\theta_T(S)\cdot\nu(\mathcal{F})$ for any finite family $\mathcal{F}$ of translates of a centrally symmetric convex body S in $\mathbb{R}^d$. Schmidt [29] showed that, for any centrally symmetric convex body S, $\delta_L(S) = \Omega(d/2^d)$; see also [5, p. 12]. Hence (3) implies the bound $\tau(\mathcal{F}) \le O(4^d/d)\theta_L(S)\cdot\nu(\mathcal{F})$. Note that $\theta_T(S) \le \theta_L(S)$. So (4) may be also better than (3) for high dimensions. Table 1 summarizes the current best upper bounds on $\alpha(C)$ and $\beta(C)$ (obtained by us and by others) for various types of convex bodies C in $\mathbb{R}^d$.

Convex body C in $\mathbb{R}^d$		*$\alpha(C)$ upper*	
arbitrary	$d=2$	6	T1
centr. symm.	$d=2$	4	T3
arbitrary	$d=3$	48	T1
centr. symm.	$d=3$	24	T2
arbitrary	$d>3$	$\min\{d(d+1)^{d-1},\ \frac{2^d}{d+1}3^{d+1}\theta_T(C)\}$	T1 T4-L2
centr. symm.	$d>3$	$\min\{d(d+1)^{d-1},\ 2^d\frac{\theta_L(C)}{\delta_L(C)},\ 5^d,\ 3^d\theta_T(C)\}$	T1 T2 T4-L2
parallelepiped	$d\ge 2$	2^{d-1}	T1-C1
Convex body C in $\mathbb{R}^d$		*$\beta(C)$ upper*	
arbitrary	$d=2$	16	[20]
centr. symm.	$d=2$	7	T4
arbitrary	$d=3$	216	†T4-L2
centr. symm.	$d=3$	125	†T4-L2
arbitrary	$d>3$	$\min\{(2d)^d,\ \frac{2^d}{d+1}3^{d+1}\theta_T(C)\}$	T4-L2
centr. symm.	$d>3$	$\min\{5^d,\ 3^d\theta_T(C)\}$	T4-L2
parallelepiped	$d\ge 2$	2^d	T4-C2

Table 1: Upper bounds on $\alpha(C)$ and $\beta(C)$ for a convex body C in $\mathbb{R}^d$. †By Theorem 4 and Lemma 2: for $d=3$, $(2d)^d = 216$ and $5^d = 125$.

A natural question is whether $\alpha(C)$ or $\beta(C)$ need to be exponential in d. The following theorem gives a positive answer:

Theorem 5. *For any convex body C in $\mathbb{R}^d$, $\beta(C) \ge \alpha(C) \ge \frac{\theta_T(C)}{\delta_T(C)}$. In particular, if C is the unit ball B^d in $\mathbb{R}^d$, then $\beta(C) \ge \alpha(C) \ge 2^{(0.599\pm o(1))d}$ as $d\to\infty$.*

Kim et al. [20] asked whether the upper bound $\tau(\mathcal{F}) \le 3\cdot\nu(\mathcal{F})$ holds for any family $\mathcal{F}$ of translates of a centrally symmetric convex body in the plane. This upper bound, if true, is best possible because there exists a finite family $\mathcal{F}$ of congruent disks (i.e., translates of a disk) such that $\tau(\mathcal{F}) = 3\cdot\nu(\mathcal{F})$ for any $\nu(\mathcal{F}) \ge 1$ [13]; see also [20, Example 10]. On the other hand, Karasev [17] proved that $\tau(\mathcal{F}) \le 3\cdot\nu(\mathcal{F}) = 3$ for any family $\mathcal{F}$ of pairwise-intersecting translates of a convex body in the plane. Also, for any family $\mathcal{F}$ of congruent disks such that $\nu(\mathcal{F}) = 2$, Kim et al. [20] confirmed that $\tau(\mathcal{F}) \le 3\cdot\nu(\mathcal{F}) = 6$. Our Corollary 1 confirms that $\tau(\mathcal{F}) \le 2\cdot\nu(\mathcal{F})$ for any finite family $\mathcal{F}$ of translates of a parallelogram. The following theorem confirms the upper bound $\tau(\mathcal{F}) \le 3\cdot\nu(\mathcal{F})$ for another special case:

Theorem 6. *For any finite family $\mathcal{F}$ of translates of a centrally symmetric convex hexagon, $\tau(\mathcal{F}) \le 3\cdot\nu(\mathcal{F})$. Moreover, if $\nu(\mathcal{F}) = 1$, then $\tau(\mathcal{F}) \le 2$.*

A hexagon $p_1p_2p_3p_4p_5p_6$ is *affinely regular* if and only if (i) it is centrally symmetric and convex, and (ii) $\overrightarrow{p_2p_1} + \overrightarrow{p_2p_3} = \overrightarrow{p_3p_4}$; see [26, p. 38]. Note that a centrally symmetric convex hexagon is not necessarily affinely regular. Grünbaum [13] showed that $\alpha_1(C) = 2$ for any affinely regular hexagon C. Theorem 6 implies a stronger and more general result that $2 = \alpha_1(C) \le \alpha(C) \le 3$ for any centrally symmetric convex hexagon C.

We now summarize the current best bounds on $\alpha(C)$ and $\beta(C)$ for squares, triangles, and disks. For the upper bounds, Theorem 3 (i), (ii), and (iii) imply that $\alpha(C) \leq 2$ for any square C, $\alpha(C) \leq 5$ for any triangle C, and $\alpha(C) \leq 4$ for any disk C, and Theorem 4 (i), (ii), and (iii) imply that $\beta(C) \leq 4$ for any square C, $\beta(C) \leq 12$ for any triangle C, and $\beta(C) \leq 7$ for any disk C. For the lower bounds, the construction in Figure 1 implies that $\beta(C) \geq \alpha(C) \geq \frac{3}{2}$ for any square C [14]. Also recall that $\alpha_1(C) \geq 3$ and $\beta_1(C) \geq 4$ for any disk C [13]. It follows that $\alpha(C) \geq \alpha_1(C) \geq 3$ and $\beta(C) \geq \beta_1(C) \geq 4$ for any disk C. Our following theorem shows that $\alpha_1(C) \geq 3$ for any triangle C. Thus we have $\beta(C) \geq \alpha(C) \geq \alpha_1(C) \geq 3$ for any triangle C.

Theorem 7. *There exists a family $\mathcal{F}$ of nine translates of a triangle such that $\nu(\mathcal{F}) = 1$ and $\tau(\mathcal{F}) = 3$.*

Table 2 summarizes the current best bounds on $\alpha(C)$ and $\beta(C)$ for some special convex bodies C in the plane.

Special convex body C in the plane	*$\alpha(C)$ lower*		*$\alpha(C)$ upper*		*$\beta(C)$ lower*		*$\beta(C)$ upper*	
centrally symmetric convex hexagon	2	[13]	3	T6	2	[13]	7	T4
square	$\frac{3}{2}$	[14]	2	T1-C1 T3	$\frac{3}{2}$	[14]	4	T4
triangle	3	T7	5	T3	3	T7	12	T4
disk	3	[13]	4	T3	4	[13]	7	T4

Table 2: Lower and upper bounds on $\alpha(C)$ and $\beta(C)$ for some special convex bodies C in the plane.

Approximation algorithms. A computational problem related to the results of this paper is finding a minimum-cardinality point set that pierces a given set of geometric objects. This problem is NP-hard even for the special case of axis-parallel unit squares in the plane [12], and it admits a polynomial-time approximation scheme for the general case of fat objects in $\mathbb{R}^d$ [8]; see also [6] for similar approximation schemes for several related problems. These approximation schemes have very high time complexities $n^{O(1/\epsilon^d)}$, and hence are impractical. Our methods for obtaining the upper bounds in Theorems 1, 2, 3, and 4 are constructive and lead to efficient constant-factor approximation algorithms for piercing a set of translates or homothets of a convex body. The approximation factors, which depend on the dimension d, are the multiplicative factors in the respective upper bounds on $\tau(\mathcal{F})$ in terms of $\nu(\mathcal{F})$ in the theorems; see also Table 1 and Table 2. For instance, Theorem 1 yields a factor-6 approximation algorithm for piercing translates of a convex body in the plane, and Theorem 4 yields (in conjunction with Lemma 2) a factor-216 approximation algorithm for piercing homothets of a convex body in 3-space. Also, for any centrally symmetric convex body S in $\mathbb{R}^d$ whose optimal lattice covering is known, Theorem 2 yields a factor-$2^d \frac{\theta_L(S)}{\delta_L(S)}$ approximation algorithm for piercing translates of S.

2 Upper bound for translates of an arbitrary convex body in $\mathbb{R}^d$

In this section we prove Theorem 1. Let $\mathcal{F}$ be a finite family of translates of a convex body C in $\mathbb{R}^d$. Let P and Q be any two parallelepipeds in $\mathbb{R}^d$ that are parallel to each other, such that $P \subseteq C \subseteq Q$. Since the two values $\tau(\mathcal{F})$ and $\nu(\mathcal{F})$ are invariant under any non-singular affine transformation of C, we can assume that P and Q are axis-parallel and have edge lengths 1 and e_i, respectively, along the axis x_i, $1 \leq i \leq d$.

We first show that $\tau(\mathcal{T}) \leq \lceil e_d \rceil \cdot \nu(\mathcal{T})$ for any finite family $\mathcal{T}$ of C-translates whose corresponding P-translates intersect a common line ℓ parallel to the axis x_d. Define the *x_d-coordinate* of a C-translate as the smallest x_d-coordinate of a point in the corresponding P-translate. Set $\mathcal{T}_1 = \mathcal{T}$, let C_1 be the C-translate in $\mathcal{T}_1$ with the smallest x_d-coordinate, and let $\mathcal{S}_1$ be the subfamily of C-translates in $\mathcal{T}_1$ that intersect C_1 ($\mathcal{S}_1$

includes C_1 itself). Then, for increasing values of i, while $\mathcal{T}_i = \mathcal{T} \setminus \bigcup_{j=1}^{i-1} \mathcal{S}_j$ is not empty, let C_i be the C-translate in $\mathcal{T}_i$ with the smallest x_d-coordinate, and let $\mathcal{S}_i$ be the subfamily of C-translates in $\mathcal{T}_i$ that intersect C_i ($\mathcal{S}_i$ includes C_i itself). The iterative process ends with a partition $\mathcal{T} = \bigcup_{i=1}^{m} \mathcal{S}_i$, where $m \le \nu(\mathcal{T})$.

Denote by c_i the x_d-coordinate of C_i. Then each C-translate in the subfamily $\mathcal{S}_i$, which is contained in a Q-translate of edge length e_d along the axis x_d, has an x_d-coordinate of at least c_i and at most $c_i + e_d$, and the corresponding P-translate, whose edge length along the axis x_d is 1, contains at least one of the $\lceil e_d \rceil$ points on ℓ with x_d-coordinates $c_i + 1, \ldots, c_i + \lceil e_d \rceil$. These $\lceil e_d \rceil$ points form a piercing set for $\mathcal{S}_i$, hence $\tau(\mathcal{S}_i) \le \lceil e_d \rceil$. It follows that

$$\tau(\mathcal{T}) \le \sum_{i=1}^{m} \tau(\mathcal{S}_i) \le \lceil e_d \rceil \cdot m \le \lceil e_d \rceil \cdot \nu(\mathcal{T}). \tag{6}$$

For $(a_1, \ldots, a_{d-1}) \in \mathbb{R}^{d-1}$, denote by $\ell(a_1, \ldots, a_{d-1})$ the following line in $\mathbb{R}^d$ that is parallel to the axis x_d:

$$\ell(a_1, \ldots, a_{d-1}) := \{ (x_1, \ldots, x_d) \mid (x_1, \ldots, x_{d-1}) = (a_1, \ldots, a_{d-1}) \}.$$

Now consider the following (infinite) set $\mathcal{L}$ of parallel lines:

$$\mathcal{L} := \{ \ell(j_1 + b_1, \ldots, j_{d-1} + b_{d-1}) \mid (j_1, \ldots, j_{d-1}) \in \mathbb{Z}^{d-1} \},$$

where $(b_1, \ldots, b_{d-1}) \in \mathbb{R}^{d-1}$ is chosen such that no line in $\mathcal{L}$ is tangent to the P-translate of any C-translate in $\mathcal{F}$. Recall that P and Q are axis-parallel and have edge lengths 1 and e_i, respectively, along the axis x_i, $1 \le i \le d$. So we have the following two properties:

1. For any C-translate in $\mathcal{F}$, the corresponding P-translate intersects exactly one line in $\mathcal{L}$.

2. For any two C-translates in $\mathcal{F}$, if the two corresponding P-translates intersect two different lines in $\mathcal{L}$ of distance at least $e_i + 1$ along some axis x_i, $1 \le i \le d-1$, then the two C-translates are disjoint.

Partition $\mathcal{F}$ into subfamilies $\mathcal{F}(j_1, \ldots, j_{d-1})$ of C-translates whose corresponding P-translates intersect a common line $\ell(j_1 + b_1, \ldots, j_{d-1} + b_{d-1})$. Let $\mathcal{F}'(k_1, \ldots, k_{d-1})$, $0 \le k_i \le \lceil e_i \rceil$, be the union of the families $\mathcal{F}(j_1, \ldots, j_{d-1})$ such that $j_i \bmod \lceil e_i + 1 \rceil = k_i$ for $1 \le i \le d-1$. It follows from (6) that the transversal number of each subfamily $\mathcal{F}'(k_1, \ldots, k_{d-1})$ is at most $\lceil e_d \rceil$ times its packing number. Therefore we have

$$\begin{aligned} \tau(\mathcal{F}) \le \sum_{(k_1,\ldots,k_{d-1})} \tau\left(\mathcal{F}'(k_1, \ldots, k_{d-1})\right) &\le \sum_{(k_1,\ldots,k_{d-1})} \lceil e_d \rceil \cdot \nu\left(\mathcal{F}'(k_1, \ldots, k_{d-1})\right) \\ &\le \sum_{(k_1,\ldots,k_{d-1})} \lceil e_d \rceil \cdot \nu(\mathcal{F}) = \left(\lceil e_d \rceil \prod_{i=1}^{d-1} \lceil e_i + 1 \rceil \right) \cdot \nu(\mathcal{F}). \end{aligned} \tag{7}$$

Since (7) holds for any pair of parallelepipeds P and Q in $\mathbb{R}^d$ that are parallel to each other and satisfy $P \subseteq C \subseteq Q$, it follows by the definition of $\gamma(C)$ in (1) that $\tau(\mathcal{F}) \le \gamma(C) \cdot \nu(\mathcal{F})$. By Lemma 1, there indeed exist two such parallelepipeds P and Q with length ratios $e_i = \lambda_i(P, Q) = d$ for $1 \le i \le d$. It then follows that $\gamma(C) \le d(d+1)^{d-1}$ for any convex body C in $\mathbb{R}^d$. This completes the proof of Theorem 1.

3 Upper bound for translates of a centrally symmetric convex body in $\mathbb{R}^d$

In this section we prove Theorem 2. Recall that $|C|$ is the Lebesgue measure of a convex body C in $\mathbb{R}^d$, and that $|\mathcal{F}|$ is the Lebesgue measure of the union of a family $\mathcal{F}$ of convex bodies in $\mathbb{R}^d$. To establish the desired bound on $\tau(\mathcal{F})$ in terms of $\nu(\mathcal{F})$ for any finite family $\mathcal{F}$ of translates of a centrally symmetric convex body S in $\mathbb{R}^d$, we link both $\tau(\mathcal{F})$ and $\nu(\mathcal{F})$ to the ratio $|\mathcal{F}|/|S|$. We first prove a lemma that links the transversal number $\tau(\mathcal{F})$ to the ratio $|\mathcal{F}|/|S|$ via the lattice covering density of S:

Lemma 3. *Let $\mathcal{F}$ be a finite family of translates of a centrally symmetric convex body S in $\mathbb{R}^d$. If there is a lattice covering of $\mathbb{R}^d$ with translates of S whose covering density is θ, $\theta \geq 1$, then $\tau(\mathcal{F}) \leq \theta \cdot |\mathcal{F}|/|S|$.*

Proof. Denote by S_p a translate of the convex body S centered at a point p. Since S is centrally symmetric, for any two points p and q, p intersects S_q if and only if q intersects S_p. Given a lattice covering of $\mathbb{R}^d$ with translates of S, every point $p \in \mathbb{R}^d$ is contained in some translate S_q in the lattice covering, hence every translate S_p contains some lattice point q.

Let Λ be a lattice such that the corresponding lattice covering with translates of S has a covering density of θ. Divide the union of the convex bodies in $\mathcal{F}$ into pieces by the cells of the lattice Λ, then translate all cells (and the pieces) to a particular cell, say σ. By the pigeonhole principle, there exists a point in σ, say p, that is covered at most $\lfloor |\mathcal{F}|/|\sigma| \rfloor$ times by the overlapping pieces of the union. Let k be the number of times that p is covered by the pieces. Now fix $\mathcal{F}$ but translate the lattice Λ to Λ' until p becomes a lattice point of Λ'. Then exactly k lattice points of Λ' are covered by the S-translates in $\mathcal{F}$. Since every S-translate in $\mathcal{F}$ contains some lattice point of Λ', we have obtained a transversal of $\mathcal{F}$ consisting of $k \leq \lfloor |\mathcal{F}|/|\sigma| \rfloor$ lattice points of Λ'. Note that $\theta = |S|/|\sigma|$, and the proof is complete. □

The following lemma[2] is a dual of the previous lemma, and links the packing number $\nu(\mathcal{F})$ to the ratio $|\mathcal{F}|/|S|$ via the lattice packing density of S:

Lemma 4. *Let $\mathcal{F}$ be a finite family of translates of a centrally symmetric convex body S in $\mathbb{R}^d$. If there is a lattice packing in $\mathbb{R}^d$ with translates of S whose packing density is δ, $\delta \leq 1$, then $\nu(\mathcal{F}) \geq \frac{\delta}{2^d} \cdot |\mathcal{F}|/|S|$.*

Proof. Let S' be a homothet of S scaled up by a factor of 2. Since S is centrally symmetric, an S-translate is contained by an S'-translate if and only if the S-translate contains the center of the S'-translate. Given a lattice packing in $\mathbb{R}^d$ with translates of S', two S'-translates centered at two different lattice points are disjoint, hence two S-translates containing two different lattice points are disjoint.

Let Λ be a lattice such that the corresponding lattice packing with translates of S' has a packing density of δ (such a lattice exists because S' is homothetic to S). Divide the union of the convex bodies in $\mathcal{F}$ into pieces by the cells of the lattice Λ, then translate all cells (and the pieces) to a particular cell, say σ. By the pigeonhole principle, there exists a point in σ, say p, that is covered at least $\lceil |\mathcal{F}|/|\sigma| \rceil$ times by the overlapping pieces of the union. Let k be the number of times that p is covered by the pieces. Now fix $\mathcal{F}$ but translate the lattice Λ to Λ' until p becomes a lattice point of Λ'. Then exactly k lattice points of Λ' are covered by the S-translates in $\mathcal{F}$. Choose k translates in $\mathcal{F}$, each containing a distinct lattice point of Λ'. Since any two S-translates containing two different lattice points of Λ' are disjoint, we have obtained a subset of $k \geq \lceil |\mathcal{F}|/|\sigma| \rceil$ pairwise-disjoint S-translates in $\mathcal{F}$. Note that $\delta = |S'|/|\sigma| = 2^d |S|/|\sigma|$, and the proof is complete. □

By Lemma 3 and Lemma 4 we have, for any finite family $\mathcal{F}$ of translates of a centrally symmetric convex body in $\mathbb{R}^d$,

$$\tau(\mathcal{F}) \leq \theta_L(S) \cdot \frac{|\mathcal{F}|}{|S|} = 2^d \cdot \frac{\theta_L(S)}{\delta_L(S)} \cdot \frac{\delta_L(S)}{2^d} \cdot \frac{|\mathcal{F}|}{|S|} \leq 2^d \cdot \frac{\theta_L(S)}{\delta_L(S)} \cdot \nu(\mathcal{F}).$$

Smith [30] proved that, for any centrally symmetric convex body S in 3-space, $\theta_L(S) \leq 3 \cdot \delta_L(S)$. This immediately implies that, for any finite family $\mathcal{F}$ of translates of a centrally symmetric convex body S in 3-space, $\tau(\mathcal{F}) \leq 2^3 \cdot 3 \cdot \nu(\mathcal{F}) = 24 \cdot \nu(\mathcal{F})$. A similar inequality for the planar case was proved by Kuperberg [21]: for any (not necessarily centrally symmetric) convex body C in the plane, $\theta(C) \leq \frac{4}{3} \cdot \delta(C)$. However, this result is not about lattice covering and packing, so we cannot use it to obtain the bound in Theorem 2 for the planar case. Instead, we prove the following "sandwich" lemma:

[2] The planar case of Lemma 4 is also implied by [4, Theorem 5].

Lemma 5. *Let $\mathcal{F}$ be a finite family of translates of a (not necessarily centrally symmetric) convex body C in $\mathbb{R}^d$. Let A and B be two centrally symmetric convex bodies in $\mathbb{R}^d$ such that $A \subseteq C \subseteq B$. Then*

$$\tau(\mathcal{F}) \le 2^d \cdot \frac{|B|}{|A|} \cdot \frac{\theta_L(A)}{\delta_L(B)} \cdot \nu(\mathcal{F}).$$

Proof. Since $A \subseteq C$, it follows by Lemma 3 that

$$\tau(\mathcal{F}) \le \theta_L(A) \cdot \frac{|\mathcal{F}|}{|A|}.$$

Since $C \subseteq B$, it follows by Lemma 4 that

$$\nu(\mathcal{F}) \ge \frac{\delta_L(B)}{2^d} \cdot \frac{|\mathcal{F}|}{|B|}.$$

Putting these together yields

$$\tau(\mathcal{F}) \le \theta_L(A) \cdot \frac{|\mathcal{F}|}{|A|} = 2^d \cdot \frac{|B|}{|A|} \cdot \frac{\theta_L(A)}{\delta_L(B)} \cdot \frac{\delta_L(B)}{2^d} \cdot \frac{|\mathcal{F}|}{|B|} \le 2^d \cdot \frac{|B|}{|A|} \cdot \frac{\theta_L(A)}{\delta_L(B)} \cdot \nu(\mathcal{F}). \qquad \square$$

We also need the following lemma which is now folklore [26, Theorem 2.5 and Theorem 2.8]:

Lemma 6. *For any centrally symmetric convex body S in the plane, there are two centrally symmetric convex hexagons H and H' such that $H \subseteq S \subseteq H'$ and $|H|/|H'| \ge 3/4$.*

Note that $\theta_L(H) = \delta_L(H) = 1$ for a centrally symmetric convex hexagon H. Set $A = H$, $B = H'$, and $C = S$ in the previous two lemmas, and we have, for any finite family $\mathcal{F}$ of translates of a centrally symmetric convex body in the plane,

$$\tau(\mathcal{F}) \le 2^2 \cdot \frac{4}{3} \cdot \frac{1}{1} \cdot \nu(\mathcal{F}) = \frac{16}{3} \cdot \nu(\mathcal{F}).$$

This completes the proof of Theorem 2.

4 Upper bound by greedy decomposition and lower bound by packing and covering

In this section we prove Theorems 3, 4, and 5.

Proof of Theorem 3. Let $\mathcal{F}$ be a finite family of translates of a convex body C in $\mathbb{R}^d$. Without loss of generality, assume that the factor $\kappa((C-C)\cap L, C)$ in (4) is minimized when $L = \{(x_1,\ldots,x_d) \mid x_d \ge 0\}$. Since $C - C$ is centrally symmetric, we have $\kappa((C-C)\cap L, C) = \kappa((C-C)\cap -L, -C)$, where $-L = \{(x_1,\ldots,x_d) \mid x_d \le 0\}$. Perform a *greedy decomposition* as follows. For $i = 1, 2, \ldots$, while $\mathcal{T}_i = \mathcal{F} \setminus \bigcup_{j=1}^{i-1} \mathcal{S}_j$ is not empty, let C_i be the translate of C in $\mathcal{T}_i$ that contains a point of the largest x_d-coordinate, and let $\mathcal{S}_i$ be the subfamily of translates in $\mathcal{T}_i$ that intersect C_i ($\mathcal{S}_i$ includes C_i itself). The iterative process ends with a partition $\mathcal{F} = \bigcup_{i=1}^m \mathcal{S}_i$, where $m \le \nu(\mathcal{F})$.

We first prove the following lemma:

Lemma 7. *Let C be a convex body and let a and b be two points in $\mathbb{R}^d$.* (i) *$C + a$ contains b if and only if $-C + b$ contains a.* (ii) *If $C + a$ intersects $C + b$, then a is contained in a translate of $C - C$ centered at b.*

Proof. (i) $b \in C + a \iff b - a \in C \iff a - b \in -C \iff a \in -C + b$. (ii) Let $z \in (C+a) \cap (C+b)$. Then $a = z - z + a \in (C+b) - (C+a) + a = C - C + b$. □

By Lemma 7 (i), piercing the translates of C in $\mathcal{S}_i$ is equivalent to covering their reference points by translates of $-C$. By Lemma 7 (ii), the reference point of each translate of C in $\mathcal{S}_i$ is contained in a translate of $C - C$ centered at the reference point of C_i. Indeed, by our choice of C_i, the reference points of all translates of C in $\mathcal{S}_i$ are contained in a translate of $(C - C) \cap -L$. Therefore we have

$$\tau(\mathcal{S}_i) \leq \kappa((C - C) \cap -L, -C) = \kappa((C - C) \cap L, C). \tag{8}$$

It follows that

$$\tau(\mathcal{F}) \leq \sum_{i=1}^{m} \tau(\mathcal{S}_i) \leq \kappa((C - C) \cap L, C) \cdot m \leq \kappa((C - C) \cap L, C) \cdot \nu(\mathcal{F}).$$

This proves the general case of Theorem 3.

In the special case that C is a centrally symmetric convex body in the plane, $C - C$ is a translate of $2C$. We have the following lemma[3] on covering $2C$ with translates of C:

Lemma 8. *Let C be a centrally symmetric convex body in the plane. Then $2C$ can be covered by seven translates of C, including one translate concentric with $2C$ and six others centered at the six vertices, respectively, of an affinely regular hexagon H_C concentric with $2C$.*

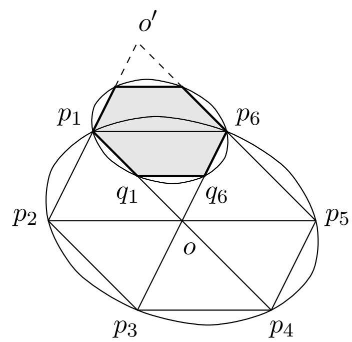

Figure 2: Covering $2C$ with seven translates of C. $2H = p_1p_2p_3p_4p_5p_6$ is an affinely regular hexagon inscribed in $2C$; o is the center of $2C$; o' is the intersection of the two lines extending p_2p_1 and p_5p_6; q_1 and q_6 are the midpoints of op_1 and op_6, respectively.

Proof. Refer to Figure 2. Assume that the center o of the convex body C is the origin. Then $2C$ is also centered at o. Let p_2 and p_5 be the intersections of the boundary of $2C$ and an arbitrary line ℓ through the origin. Choose two points p_1 and p_6 on the boundary of $2C$ on one side of the line ℓ, and choose two points p_3 and p_4 on the other side, such that $\overrightarrow{p_1p_6} = \overrightarrow{p_3p_4} = \frac{1}{2}\overrightarrow{p_2p_5}$. Then $p_1p_2p_3p_4p_5p_6$ is an affinely regular hexagon. Let $2H$ be this hexagon inscribed in $2C$. Consider the (shaded) hexagon H' that is a translate of H with two opposite vertices p_1 and p_6. Let q_1 and q_6 be the midpoints of op_1 and op_6, respectively. Then q_1 and q_6 are also vertices of H'. The two hexagons $2H$ and H' are homothetic with ratio 2 and with homothety center at the intersection o' of the two lines extending p_2p_1 and p_5p_6. Let C' be a translate of C such that H' is inscribed in C'. Then C' covers the part of $2C$ between the two rays $\overrightarrow{op_1}$ and $\overrightarrow{op_6}$. It follows that $2C$ is covered by seven translates of C, one centered at the origin, and six others centered at the midpoints of the six sides of $2H$, respectively. The six midpoints are clearly the vertices of another (smaller) affinely regular hexagon concentric with $2C$. Let H_C be this hexagon, and the proof is complete. □

[3] This lemma is implicit in a result by Grünbaum [13, Theorem 4]. Nevertheless we present our own simple proof here for completeness.

Choose the halfplane L through the center of $2C$ and any two opposite vertices of the hexagon $2H = p_1p_2p_3p_4p_5p_6$ in Lemma 8. Then $\kappa((C - C) \cap L, C) \leq 4$. It follows that $\tau(\mathcal{F}) \leq 4 \cdot \nu(\mathcal{F})$ for any finite family $\mathcal{F}$ of translates of a centrally symmetric convex body in the plane.

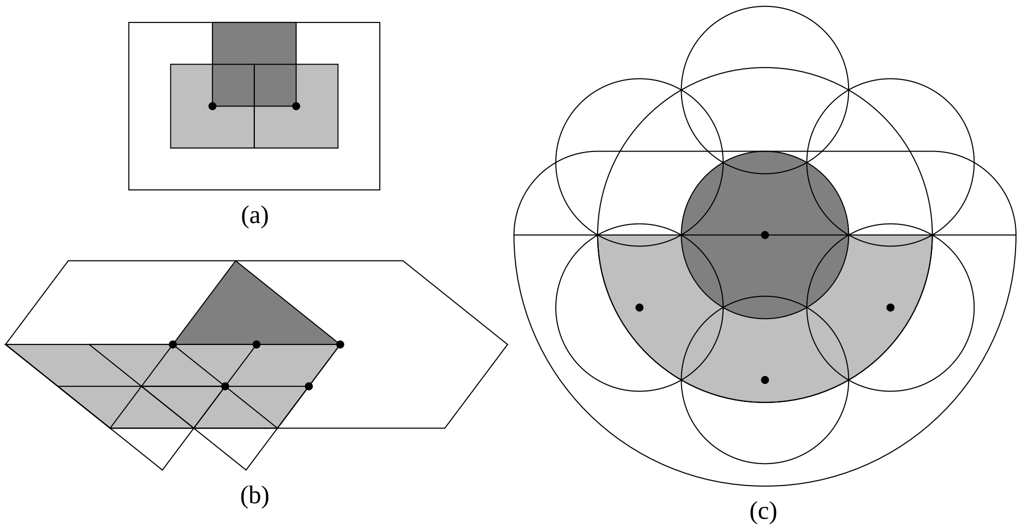

Figure 3: Piercing a subfamily $\mathcal{S}_i$ of translates that intersect the highest translate C_i (dark-shaded). (a) The centers of the squares are contained in the light-shaded rectangle; the squares can be pierced by two points. (b) The lower-left vertices of the triangles are contained in the light-shaded trapezoid; the triangles can be pierced by five points. (c) The centers of the disks are contained in the light-shaded half-disk; the disks can be pierced by four points.

To complete the proof of Theorem 3, we apply the greedy decomposition algorithm to some simple types of convex bodies in the plane: squares, triangles, and disks. We use some known bounds on $\tau(\mathcal{F})$ for families $\mathcal{F}$ with small $\nu(\mathcal{F})$, for example, $\alpha_1(C)$ for $\nu(\mathcal{F}) = 1$, to obtain slightly better upper bounds for these special cases. We refer to Figure 3, where the x_1 and x_2 axes are the horizontal and vertical axes.

First let C be a square, and refer to Figure 3 (a). Corollary 1 implies that $\tau(\mathcal{F}) \leq 2 \cdot \nu(\mathcal{F})$ for any finite family $\mathcal{F}$ of translates of C. We obtain a slightly better bound by a tighter analysis of the greedy decomposition algorithm. Assume that C is axis-parallel and has side length 1. Then the centers of the squares in $\mathcal{S}_i$ are contained in the light-shaded rectangle of width 2 and height 1, which is covered by two unit squares centered at the two lower vertices of C_i. Each square in $\mathcal{S}_i$ contains one of the two lower vertices of C_i, thus $\tau(\mathcal{S}_i) \leq 2$. Consider two cases:

1. $m \leq \nu(\mathcal{F}) - 1$. Then

$$\tau(\mathcal{F}) \leq \sum_{i=1}^{m} \tau(\mathcal{S}_i) \leq 2 \cdot (\nu(\mathcal{F}) - 1) = 2 \cdot \nu(\mathcal{F}) - 2.$$

2. $m = \nu(\mathcal{F})$. Then $\nu(\mathcal{S}_m) = 1$. It follows that $\tau(\mathcal{S}_m) \leq \alpha_1(C) = 1$ [13]. Then

$$\tau(\mathcal{F}) \leq \sum_{i=1}^{m} \tau(\mathcal{S}_i) \leq 2 \cdot (\nu(\mathcal{F}) - 1) + 1 = 2 \cdot \nu(\mathcal{F}) - 1.$$

Next let C be a triangle, and refer to Figure 3 (b). Assume that C has a horizontal lower side. The lower-left vertices of the triangles in $\mathcal{S}_i$ are contained in the light-shaded trapezoid, which can be covered by five translates of $-C$. Hence each triangle in $\mathcal{S}_i$ contains one of the upper-right vertices of these five translates, thus $\tau(\mathcal{S}_i) \leq 5$. The proof can be finished in the same way as for squares by considering the two cases $m \leq \nu(\mathcal{F}) - 1$ and $m = \nu(\mathcal{F})$, and using the fact that $\alpha_1(C) = 3$ for any triangle C [7].

Finally let C be a disk, and refer to Figure 3 (c). Assume that C has radius 1. Then the centers of the disks in $\mathcal{S}_i$ are contained in the light-shaded half-disk of radius 2. By Lemma 8, a disk of radius 2 can be covered by seven disks of radius 1, with one disk in the middle and six others around in a hexagonal formation. Therefore the half-disk of radius 2 can be covered by four disks of radius 1. The center of each disk in $\mathcal{S}_i$ is contained by one of the four disks; by symmetry, each disk in $\mathcal{S}_i$ contains the center of one of the four disks, thus $\tau(\mathcal{S}_i) \leq 4$. Again, the proof can be finished by considering the two cases $m \leq \nu(\mathcal{F}) - 1$ and $m = \nu(\mathcal{F})$ as done for squares and triangles, and using the fact that $\alpha_1(C) = 3$ for any disk C [13]. Indeed the same argument shows that $\tau(\mathcal{F}) \leq 4 \cdot (\nu(\mathcal{F}) - 1) + 3 = 4 \cdot \nu(\mathcal{F}) - 1$ for any centrally symmetric convex body C in the plane since $\alpha_1(C) \leq 3$ also holds [17]. This completes the proof of Theorem 3.

Proof of Theorem 4. Let $\mathcal{F}$ be a finite family of homothets of a convex body C in $\mathbb{R}^d$. We again use greedy decomposition as in the proof of Theorem 3. The only difference in the algorithm is that C_i is now chosen as the smallest homothet of C in $\mathcal{T}_i$. By our choice of C_i, each homothet in $\mathcal{S}_i$ contains a translate of C_i that intersects C_i. Hence the bound $\tau(\mathcal{S}_i) \leq \kappa(C - C, C)$ follows in a similar way as the derivation of (8).

Let now C be a centrally symmetric convex body in the plane. By Lemma 8, we have $\kappa(C - C, C) \leq 7$. Then $\tau(\mathcal{S}_i) \leq \kappa(C - C, C) \leq 7$, from which it follows that $\tau(\mathcal{F}) \leq 7 \cdot \nu(\mathcal{F})$ for any centrally symmetric convex body C in the plane.

The analysis for special types of convex bodies C in the plane (squares, triangles, and disks) is also similar to the corresponding analysis in the proof of Theorem 3. We obtain the bound $\tau(\mathcal{S}_i) \leq \kappa(C - C, C)$ and show that $\kappa(C - C, C) \leq 4$ for any square C, $\kappa(C - C, C) \leq 12$ for any triangle C, and $\kappa(C - C, C) \leq 7$ for any disk C, then use $\beta_1(C)$ instead of $\alpha_1(C)$ to bound $\tau(\mathcal{S}_m)$ in case 2. As discussed in the introduction, it is known that $\beta_1(C) = 1$ for any square C [13], $\beta_1(C) = 3$ for any triangle C [7], and $\beta_1(C) = 4$ for any disk C [13, 9]. This completes the proof of Theorem 4.

Proof of Theorem 5. Let C be a convex body in $\mathbb{R}^d$ and n be a positive integer. We will show that $\beta(C) \geq \alpha(C) \geq \theta_T(C)/\delta_T(C)$ by constructing a finite family $\mathcal{F}_n$ of n^{2d} translates of C, such that

$$\lim_{n \to \infty} \frac{\tau(\mathcal{F}_n)}{\nu(\mathcal{F}_n)} \geq \frac{\theta_T(C)}{\delta_T(C)}. \tag{9}$$

By Lemma 1, there exist two homothetic parallelepipeds P and Q with ratio d such that $P \subseteq C \subseteq Q$. Without loss of generality (via an affine transformation), we can assume that P and Q are axis-parallel hypercubes of side lengths 1 and d, respectively, and that P is centered at the origin. Let $\mathcal{F}_n$ be a family of translates of C

$$\mathcal{F}_n := \{C + t \mid t \in T_n\}$$

corresponding to a set T_n of n^{2d} regularly placed reference points

$$T_n := \{(t_1/n, \ldots, t_d/n) \mid (t_1, \ldots, t_d) \in \mathbb{Z}^d, 1 \leq t_1, \ldots, t_d \leq n^2\}.$$

Denote by $H(\ell)$ any axis-parallel hypercube of side length ℓ.

We first obtain an upper bound on $\nu(\mathcal{F}_n)$. For each $C + t \in \mathcal{F}_n$, we have $C + t \subseteq C + T_n \subseteq Q + T_n$. Note that $Q + T_n$ is an axis-parallel hypercube of side length exactly $n - \frac{1}{n} + d$. Denote by $\delta_T(X, Y)$ the supremum of the packing density of a domain $Y \subseteq \mathbb{R}^d$ by translates of X. By a volume argument, we have

$$\nu(\mathcal{F}_n) \leq \frac{\delta_T(C, Q + T_n) \cdot |Q + T_n|}{|C|} = \frac{\delta_T\big(C, H(n - \frac{1}{n} + d)\big) \cdot (n - \frac{1}{n} + d)^d}{|C|}. \tag{10}$$

We next obtain a lower bound on $\tau(\mathcal{F}_n)$. By Lemma 7 (i), piercing the family $\mathcal{F}_n$ of translates of C is equivalent to covering the corresponding set T_n of reference points by translates of $-C$. Let S_n be any set of points such that $T_n \subseteq -C + S_n$, that is, T_n is covered by the set $\{-C + s \mid s \in S_n\}$ of translates of $-C$. We also have $-\frac{1}{n}P \subseteq -\frac{1}{n}C$ since $P \subseteq C$. It follows that

$$-\tfrac{1}{n}P + T_n \subseteq -\tfrac{1}{n}C + (-C + S_n) = -(1 + \tfrac{1}{n})C + S_n.$$

Thus $\tau(\mathcal{F}_n)$ is at least the minimum number of translates of $-(1 + \frac{1}{n})C$ that cover $-\frac{1}{n}P + T_n$. Note that $-\frac{1}{n}P + T_n$ is an axis-parallel hypercube of side length exactly n. Denote by $\theta_T(X, Y)$ the infimum of the covering density of a domain $Y \subseteq \mathbb{R}^d$ by translates of X. Again by a volume argument, we have

$$\tau(\mathcal{F}_n) \geq \frac{\theta_T\big(-(1 + \frac{1}{n})C, -\frac{1}{n}P + T_n\big) \cdot |-\frac{1}{n}P + T_n|}{|-(1 + \frac{1}{n})C|} = \frac{\theta_T\big((1 + \frac{1}{n})C, H(n)\big) \cdot n^d}{(1 + \frac{1}{n})^d \cdot |C|}. \tag{11}$$

From the two inequalities (10) and (11), it follows that

$$\frac{\tau(\mathcal{F}_n)}{\nu(\mathcal{F}_n)} \geq \frac{\theta_T\big((1 + \frac{1}{n})C, H(n)\big)}{\delta_T\big(C, H(n - \frac{1}{n} + d)\big)} \cdot \frac{1}{(1 + \frac{1}{n})^d(1 - \frac{1}{n^2} + \frac{d}{n})^d}.$$

Taking the limit as $n \to \infty$, we have $\theta_T\big((1 + \frac{1}{n})C, H(n)\big) \to \theta_T(C)$, $\delta_T\big(C, H(n - \frac{1}{n} + d)\big) \to \delta_T(C)$, and $(1 + \frac{1}{n})^d(1 - \frac{1}{n^2} + \frac{d}{n})^d \to 1$. This yields (9) as desired.

We now consider the special case that C is the d-dimensional unit ball B^d in $\mathbb{R}^d$. We clearly have $\theta_T(B^d) \geq 1$. Kabatjanskiĭ and Levenšteĭn [15] showed that $\delta_T(B^d) = \delta(B^d) \leq 2^{-(0.599 \pm o(1))d}$ as $d \to \infty$; see also [5, p. 50]. Therefore we have

$$\beta(B^d) \geq \alpha(B^d) \geq \frac{\theta_T(B^d)}{\delta_T(B^d)} \geq 2^{(0.599 \pm o(1))d} \text{ as } d \to \infty.$$

This completes the proof of Theorem 5.

5 Upper bound for translates of a centrally symmetric convex hexagon

In this section we prove Theorem 6. Let $\mathcal{F}$ be a finite family of translates of a centrally symmetric convex hexagon H in the plane.

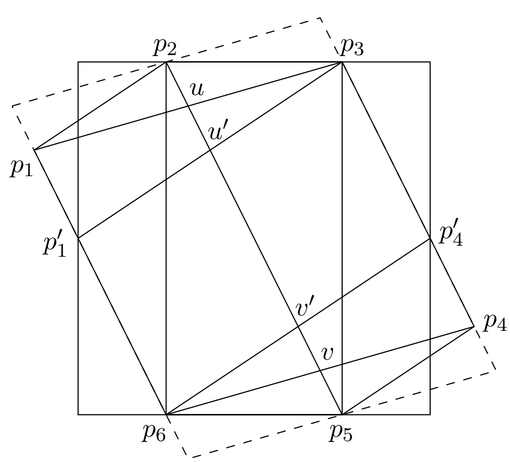

Figure 4: A centrally symmetric convex hexagon $H = p_1p_2p_3p_4p_5p_6$.

We refer to Figure 4 for the general case $\nu(\mathcal{F}) \geq 1$. We will prove that $\tau(\mathcal{F}) \leq 3 \cdot \nu(\mathcal{F})$. By Theorem 1, it suffices to show that $\gamma(H) \leq 3$. We will show that $\gamma(H) \leq 3$ by finding two parallelograms P and

Q that are parallel to each other, with length ratios $w = \lambda_1(P, Q) \le 2$ and $h = \lambda_2(P, Q) = 1$, such that $P \subseteq H \subseteq Q$. Let $H = p_1p_2p_3p_4p_5p_6$. Without loss of generality (via an affine transformation), the parallelogram $p_2p_3p_5p_6$ is an axis-parallel rectangle of width $1/2$ and height 1. If the hexagon is contained in an axis-parallel unit square, then we can choose P and Q as the rectangle $p_2p_3p_5p_6$ and the square, whose length ratios are $w = 2$ and $h = 1$. Suppose otherwise. Assume that p_1 is higher than p_4. Then we choose P as the parallelogram $p_1p_3p_4p_6$ and Q as the (dashed) parallelogram circumscribing H and parallel to P. Let u be the intersection of p_1p_3 and p_2p_5, and let v be the intersection of p_4p_6 and p_2p_5. The length ratios of P and Q are $w = |p_2p_5|/|uv|$ and $h = 1$, where w is maximized to 2 when p_1 and p_4 are the midpoints of the two vertical sides of the unit square.

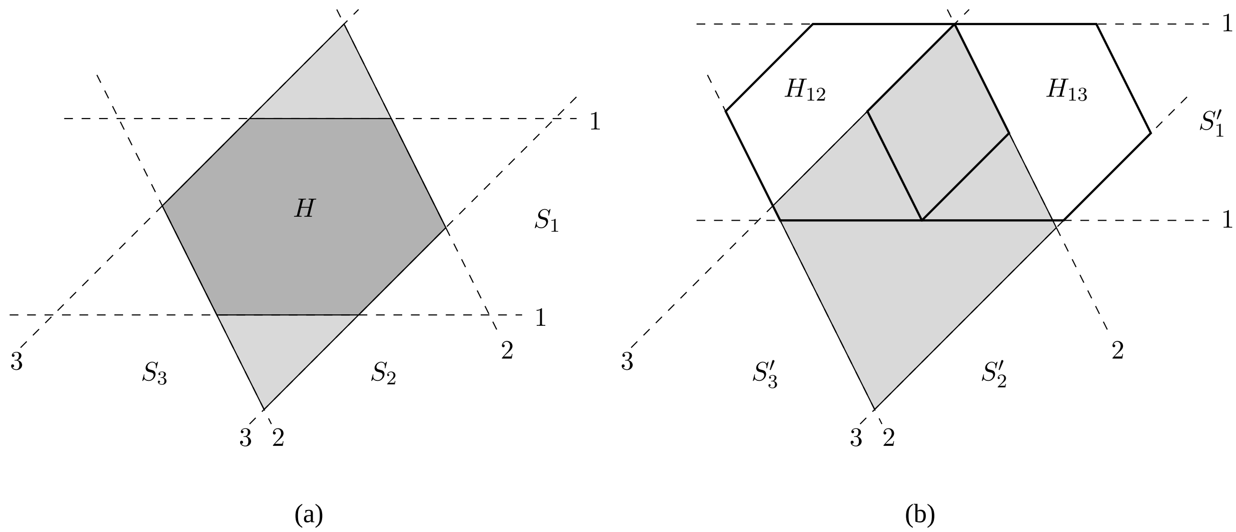

Figure 5: (a) A centrally symmetric convex hexagon H is the intersection of three strips S_1, S_2, and S_3. (b) The centers of all translates of H in $\mathcal{F}$ are contained in the intersection of three strips S'_1, S'_2, and S'_3 that are translates of S_1, S_2, and S_3, respectively. The intersection $S_1 \cap S_2 \cap S_3$ is covered by two translates of H: $H_{12} \subseteq S'_1 \cap S'_2$ and $H_{13} \subseteq S'_1 \cap S'_3$.

We refer to Figure 5 for the special case $\nu(\mathcal{F}) = 1$. We will prove that $\tau(\mathcal{F}) \le 2$. The centrally symmetric convex hexagon H is the intersection of three strips S_1, S_2, and S_3, each bounded by the two supporting lines of a pair of parallel edges of H. Without loss of generality, assume that the strip S_1 is horizontal. Let A and B be the highest and lowest translates of H in $\mathcal{F}$, respectively. Then the y-coordinates of the centers of A and B differ by at most the width of the strip S_1. This implies that the centers of all translates of H in $\mathcal{F}$ are contained in a translate of S_1. Apply the same argument to the other two strips S_2 and S_3. It follows that the centers of all translates of H in $\mathcal{F}$ are contained a convex polygon that is the intersection of three strips S'_1, S'_2, and S'_3, which are translates of S_1, S_2, and S_3, respectively. Without loss of generality, we can assume that $S'_2 = S_2$ and $S'_3 = S_3$ as illustrated in Figure 5, and further assume that the strip S'_1 is cut by the parallelogram that is the intersection of S'_2 and S'_3, i.e., $S'_1 \setminus (S'_2 \cap S'_3)$ is disconnected. Let H_{12} and H_{13} be the two unique translates of H contained in $S'_1 \cap S'_2$ and $S'_1 \cap S'_3$, respectively. Then the union of H_{12} and H_{13} is a hexagon in the strip S'_1, and $S'_1 \cap S'_2 \cap S'_3$ is contained in $H_{12} \cup H_{13}$. It follows that two points (the centers of H_{12} and H_{13}) are enough to pierce all members of $\mathcal{F}$. This completes the proof of Theorem 6.

6 Lower bound for translates of a triangle

In this section we prove Theorem 7. Chakerian and Stein [7] proved that $\tau(\mathcal{F}) \le 3$ for any family $\mathcal{F}$ of pairwise-intersecting translates of a triangle in the plane, and showed that the number 3 here is best possible by a construction using an infinite number of translates. See also [22] for a related result. We give a simpler construction using only nine translates.

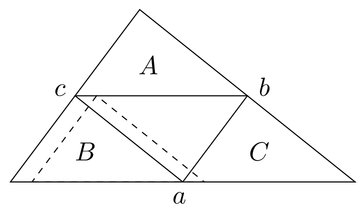

Figure 6: Three pairwise-tangent translates A, B, and C of a triangle T. The dashed triangle is B_C.

We refer to Figure 6. Let A, B, and C be three translates of a triangle T that are pairwise-tangent with intersections at three vertices a, b, c. We obtain six more translates of T as follows. Translate a copy of T for a short distance ϵ from B toward C, and let B_C be the resulting translate. Similarly obtain A_B, A_C, B_A, C_A, and C_B. Let $\mathcal{F}$ be the family of nine translates $A, B, C, A_B, A_C, B_C, B_A, C_A, C_B$. We next show that $\nu(\mathcal{F}) = 1$ and $\tau(\mathcal{F}) = 3$.

It is easy to check that any two members of $\mathcal{F}$ intersect, and that the three vertices a, b, c pierce all members of $\mathcal{F}$. Thus $\nu(\mathcal{F}) = 1$ and $\tau(\mathcal{F}) \le 3$. It remains to show that $\tau(\mathcal{F}) \ge 3$, that is, three points are necessary to pierce all members of $\mathcal{F}$. Suppose for contradiction that two points are enough. Then one of the two points must be a, b, or c since A, B, and C are pairwise-tangent. Assume that a is one of the two points. Then the other point must intersect the three translates A, B_A, and C_A that do not contain the point a. But these three translates do not have a common point when ϵ is sufficiently small. We have reached a contradiction. This completes the proof of Theorem 7.

7 Conclusion

We believe that our bounds in Lemma 3 and Lemma 4 are not tight. We have the following conjectures:

Conjecture 1. *Let $\mathcal{F}$ be a finite family of translates of a centrally symmetric convex body S in the plane. Then $\tau(\mathcal{F}) \le |\mathcal{F}|/|S|$.*

Conjecture 2. *Let $\mathcal{F}$ be a finite family of translates of a centrally symmetric convex body S in the plane. Then $\nu(\mathcal{F}) \ge \frac{1}{4} \cdot |\mathcal{F}|/|S|$.*

If both conjectures were to hold (note that they hold for the special cases when S is a parallelogram or a centrally symmetric convex hexagon since $\theta_L(S) = \delta_L(S) = 1$ in such cases), then we would have an alternative proof of essentially the same bound $\tau(\mathcal{F}) \le 4 \cdot \nu(\mathcal{F})$ as in Theorem 3 for any finite family $\mathcal{F}$ of translates of a centrally symmetric convex body in the plane. Conjecture 2 is related to another recent conjecture [3] in the spirit of Rado [27]:

Conjecture 3 (Bereg, Dumitrescu, and Jiang [3]). *For any set $\mathcal{S}$ of (not necessary congruent) closed disks in the plane, there exists a subset $\mathcal{I}$ of pairwise-disjoint disks such that $|\mathcal{I}|/|\mathcal{F}| \ge \frac{1}{4}$.*

Note that a disk D is centrally symmetric; for any finite family $\mathcal{F}$ of congruent disks (i.e., translates of a disk) in the plane, $\nu(\mathcal{F}) \ge \frac{1}{4} \cdot |\mathcal{F}|/|D|$ if and only if there exists a subset $\mathcal{I}$ of pairwise-disjoint disks such that $|\mathcal{I}|/|\mathcal{F}| \ge \frac{1}{4}$.

Note. After completion of this work and shortly before journal submission, we learned that very recently, Naszódi and Taschuk [24] independently obtained some results similar in nature to our Theorems 4 and 5. There are however differences in the specific bounds:

1. They proved[4] that $\beta(C) \leq 2^d\binom{2d}{d}(d \ln d + d \ln \ln d + 5d)$ for any convex body C in $\mathbb{R}^d$, and that $\beta(C) \leq 3^d(d \ln d + d \ln \ln d + 5d)$ for any centrally symmetric convex body C in $\mathbb{R}^d$. Note that their upper bound for the centrally symmetric case is essentially the same as our bound $\beta(C) \leq 3^d\theta_T(C)$ by Theorem 4 and Lemma 2. Their upper bound for the general case, however, is weaker than our bound $\beta(C) \leq \frac{2^d}{d+1}3^{d+1}\theta_T(C)$, also by Theorem 4 and Lemma 2. By Stirling's formula, $\binom{2d}{d} = \frac{(2d)!}{(d!)^2} = \Theta(4^d/\sqrt{d})$. Compare the factor $2^d\binom{2d}{d} = \Theta(8^d/\sqrt{d})$ in their bound with the factor $\frac{2^d}{d+1}3^{d+1} = \Theta(6^d/d)$ in our bound.

2. They also derived the following lower bound: for sufficiently large d, there is a convex body C in $\mathbb{R}^d$ such that $\alpha(C) \geq \frac{1}{2}(1.058)^d$. This lower bound is analogous to our exponential lower bound in Theorem 5: if C is the unit ball B^d in $\mathbb{R}^d$, then $\alpha(C) \geq 2^{(0.599 \pm o(1))d} \approx (1.51)^d$ as $d \to \infty$. Recall that our lower bound for the unit ball B^d follows from a general lower bound for any convex body C in $\mathbb{R}^d$, namely, $\alpha(C) \geq \frac{\theta_T(C)}{\delta_T(C)}$. A comparison shows that their lower bound is both weaker and less general than ours.

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