

Lattice spanners of low degree*

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Abstract

Let $\delta_0(P, k)$ denote the degree k dilation of a point set P in the domain of plane geometric spanners. If Λ is the infinite square lattice, it is shown that $1 + \sqrt{2} \leq \delta_0(\Lambda, 3) \leq (3 + 2\sqrt{2})5^{-1/2} = 2.6065\dots$ and $\delta_0(\Lambda, 4) = \sqrt{2}$. If Λ is the infinite hexagonal lattice, it is shown that $\delta_0(\Lambda, 3) = 1 + \sqrt{3}$ and $\delta_0(\Lambda, 4) = 2$. All our constructions are planar lattice tilings constrained to degree 3 or 4.

Keywords: geometric graph, plane spanner, vertex dilation, stretch factor, planar lattice.

1 Introduction

Let P be a (possibly infinite) set of points in the Euclidean plane. A *geometric graph* embedded on P is a graph $G = (V, E)$ where $V = P$ and an edge $uv \in E$ is the line segment connecting u and v . View G as a edge-weighted graph, where the weight of uv is the Euclidean distance between u and v . A geometric graph G is a *t -spanner*, for some $t \geq 1$, if for every pair of vertices u, v in V , the length of the shortest path $\pi_G(u, v)$ between u and v in G is at most t times $|uv|$, i.e., $\forall u, v \in V, |\pi_G(u, v)| \leq t|uv|$. Obviously, the complete geometric graph on a set of points is a 1-spanner. When there is no need to specify t , the rather imprecise term *geometric spanner* is also used. A geometric spanner G is *plane* if no two edges in G cross. Here we only consider plane geometric spanners. A geometric spanner of degree at most k is called *degree k geometric spanner*.

Consider a geometric spanner $G = (V, E)$. The *vertex dilation* or *stretch factor* of a pair $u, v \in V$, denoted $\delta_G(u, v)$, is defined as $\delta_G(u, v) = |\pi_G(u, v)|/|uv|$. If G is clear from the context, we simply write $\delta(u, v)$. The *vertex dilation* or *stretch factor* of G , denoted $\delta(G)$, is defined as $\delta(G) = \sup_{u, v \in V} \delta_G(u, v)$. The terms *graph theoretic dilation* and *spanning ratio* are also used [17, 22, 30].

Given a point set P , let the *dilation* of P , denoted by $\delta_0(P)$, be the minimum stretch factor of a plane geometric graph (equivalently, triangulation) on vertex set P ; see [29]. Similarly, let the *degree k dilation* of P , denoted by $\delta_0(P, k)$, be the minimum stretch factor of a plane geometric graph of degree at most k on vertex set P . Clearly, $\delta_0(P, k) \geq \delta_0(P)$ holds for any k . Furthermore, $\delta_0(P, j) \geq \delta_0(P, k)$ holds for any $j < k$. (Note that the term *dilation* has been also used with different meanings in the literature, see for instance [8, 23].)

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The field of geometric spanners has witnessed a great deal of interest from researchers, both in theory and applications; see for instance the survey articles [8, 19, 20, 30]. For the current status of various open problems in this area, the reader is referred to the web-page maintained by Smid [31].

Typical objectives include constructions of low stretch factor geometric spanners that have few edges, bounded degree, low weight and/or diameter, etc. Geometric spanners find their applications in the areas of robotics, computer networks, distributed systems and many others. Various algorithmic and structural results on sparse geometric spanners can be found in [1, 2, 3, 10, 11, 18, 23, 25].

Chew [12] was the first to show that it is always possible to construct a plane 2-spanner with $O(n)$ edges on a set of n points; more recently, Xia [32] proved a slightly sharper upper bound of 1.998 using Delaunay triangulations. Bose et al. [7] showed that there exists a plane t -spanner of degree at most 27 on any set of points in the Euclidean plane where $t \approx 10.02$. The result was subsequently improved in [4, 9, 6, 21, 26] in terms of degree. Recently, Bonichon et al. [5] reduced the degree to 4 with $t \approx 156.82$. The question whether the degree can be reduced to 3 remains open at the time of this writing; if one does not insist on having a plane spanner, Das et al. [13] showed that degree 3 is achievable. From the other direction, lower bounds on the stretch factors of plane spanners for finite point sets have been investigated in [15, 23, 29].

It is natural to study the existence of low-degree spanners of fundamental regular structures, such as point lattices. Indeed, these have been the focus of interest since the early days of computing. One such intense research area concerns VLSI [24]. Other applications of spanners (not necessarily geometric) are in the areas of computer networks and parallel computing; see for instance [27, 28]. While the authors of [27, 28] do examine grid structures (including planar ones), the resulting stretch factors however are not defined (or measured) in geometric terms. More recently, lattice structures at a larger scale are used in industrial design, modern urban design and outer space design. Indeed, Manhattan-like layout of facilities and road connections are very convenient to plan and deploy, frequently in an automatic manner. Studying the stretch factors that can be achieved in low degree spanners of point sets with a lattice structure appears to be quite useful. The two most common lattices are the square lattice and the hexagonal lattice.

According to an argument due to Das and Heffernan [13],[30, p. 468], the n points in a $\sqrt{n} \times \sqrt{n}$ section of the integer lattice cannot be connected in a path or cycle with stretch factor $o(\sqrt{n})$, $O(1)$ in particular. Similarly, no degree 2 plane spanner of the infinite integer lattice can have stretch factor $O(1)$, hence a minimum degree of 3 is necessary in achieving a constant stretch factor. The same facts hold for the infinite hexagonal lattice.

Our results. Let Λ be the infinite square lattice. We show that the degree 3 and 4 dilation of this lattice are bounded as follows:

- (i) $1 + \sqrt{2} \leq \delta_0(\Lambda, 3) \leq (3 + 2\sqrt{2}) 5^{-1/2}$ (Theorem 1, Section 3).
- (ii) $\delta_0(\Lambda, 4) = \sqrt{2}$ (Theorem 2, Section 3).

If Λ is the infinite hexagonal lattice, we show that

- (i) $\delta_0(\Lambda, 3) = 1 + \sqrt{3}$ (Theorem 3, Section 4).
- (ii) $\delta_0(\Lambda, 4) = 2$ (Theorem 4, Section 4).

2 Preliminaries

By the well known Cauchy-Schwarz inequality for $n = 2$, if $a, b, x, y \in \mathbf{R}^+$, then

$$g(x, y) = \frac{ax + by}{\sqrt{x^2 + y^2}} \leq \sqrt{a^2 + b^2},$$

and moreover, $g(x, y) = \sqrt{a^2 + b^2}$ when $x/y = a/b$. In this paper, we will use this inequality in an equivalent form:

Fact 1. *Let $a, b, \lambda \in \mathbf{R}^+$. Then $f(\lambda) = \frac{a\lambda + b}{\sqrt{\lambda^2 + 1}} \leq \sqrt{a^2 + b^2}$, and moreover, $f(\lambda) = \sqrt{a^2 + b^2}$ when $\lambda = a/b$.*

Notations and assumptions. Let P be a planar point set and $G = (V, E)$ be a plane geometric graph on vertex set P . For $p, q \in P$, pq denotes the connecting segment and $|pq|$ denotes its Euclidean length. The degree of a vertex (point) $p \in V$ is denoted by $\deg(p)$. For a specific point set $P = \{p_1, \dots, p_n\}$, we denote the shortest path between p_s, p_t in G consisting of vertices in the order p_s, \dots, p_t using $\rho(p_s, \dots, p_t)$ and by $|\rho(p_s, \dots, p_t)|$ its total Euclidean length. The graphs we construct have the property that no edge contains a point of P in its interior.

3 The square lattice

This section is devoted to the degree 3 and 4 dilation of the square lattice. In [16], we showed that the degree 3 dilation of the infinite square lattice is at most $(7 + 5\sqrt{2})29^{-1/2} = 2.6129\dots$ Here we improve this upper bound to $\delta_0 := (3 + 2\sqrt{2})5^{-1/2} = 2.6065\dots$ We believe that this upper bound is the best possible, and so in this section we present two degree 3 spanners for the infinite square lattice that attain this bound. Another possible candidate is presented in Section 5.

Theorem 1. *Let Λ be the infinite square lattice. Then,*

$$2.4142\dots = 1 + \sqrt{2} \leq \delta_0(\Lambda, 3) \leq (2\sqrt{2} + 3)5^{-1/2} = 2.6065\dots$$

Proof. To prove the lower bound, consider any point $p_0 \in \Lambda$ and its eight neighbors p_1, \dots, p_8 , as in Fig. 1. Since $\deg(p_0) \leq 3$, p_0 can be connected to at most three neighbors from $\{p_2, p_4, p_6, p_8\}$.

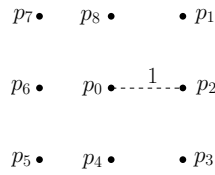


Figure 1: Illustrating the lower bound of $1 + \sqrt{2}$ for the square lattice.

We may assume that the edge p_0p_2 is not present; then

$$\delta(p_0, p_2) \geq \frac{|\rho(p_0, p_i, p_2)|}{|p_0p_2|} \geq 1 + \sqrt{2}, \text{ where } i \in \{1, 3, 4, 8\}.$$

To prove the upper bound, we construct a plane degree 3 geometric graph G as illustrated in Fig. 2 (left); observe that there are four types of vertices in G . For any two lattice points $p, q \in \Lambda$, we

construct a path in G . Set $p = (0, 0)$ as the origin and consider the four quadrants W_i , $i = 1, \dots, 4$, labeled counterclockwise in the standard fashion; see Fig. 2 (right). Points on the dividing lines are assigned arbitrarily to any of the two adjacent quadrants. By the symmetry of G , we can assume that q lies in the first quadrant, thus $q = (x, y)$, where $x, y \geq 0$, while the origin $p = (0, 0)$ can be at any of the four possible types of lattice points.

Consider the path from $p = (0, 0)$ to $q = (x, y)$ via (z, z) , where $z = \min(x, y)$, that visits every other lattice point on this diagonal segment as shown in Fig. 3, and let $\ell(x, y)$ denote its length. If $x = 0$, the stretch factor is easily seen to be at most $1 + \sqrt{2}$. Since a horizontal path connecting two points with the same y -coordinate at distance a is always shorter than any path connecting two points with the same x -coordinate at the same distance a , it is enough to prove our bound on the stretch factor in the case $y \geq x$ (i.e., $z = x$). We thus subsequently assume that $y \geq x \geq 1$.

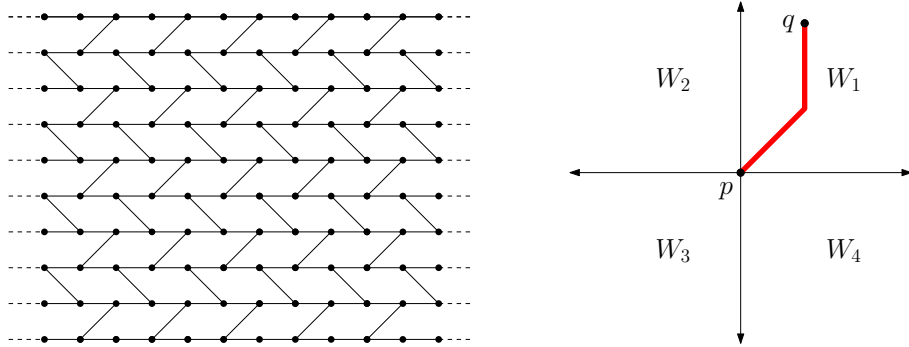


Figure 2: Left: a degree 3 plane graph on Λ . Right: a schematic diagram showing the path between p, q (when $x \leq y$). The bold path consist of segments of lengths 1 and $\sqrt{2}$.

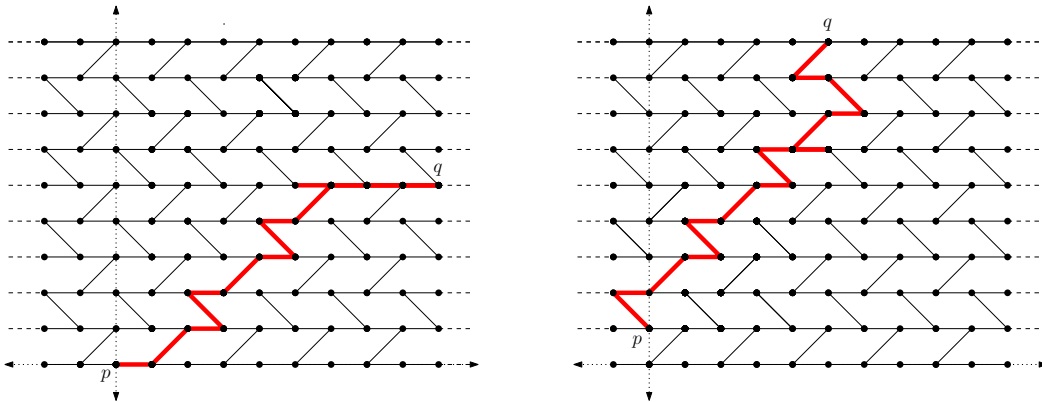


Figure 3: Paths connecting p to q in G generated by the procedure outlined in the text. Observe that in both examples a unit horizontal edge is traversed in both directions (but can be shortcut).

Observe that connecting points (a, a) with $(a + 2, a + 2)$, for any $a \geq 0$, requires length $2 + 2\sqrt{2}$, and that connecting points (a, a) with $(a + 1, a + 1)$, for any $a \geq 0$, requires length at most $2 + \sqrt{2}$. It follows that

$$\begin{aligned} \ell(x, y) &\leq \left(2 \left\lceil \frac{x}{2} \right\rceil + \sqrt{2}x\right) + (y - x)(1 + \sqrt{2}) \\ &\leq 2 \left(\frac{x + 1}{2} \right) + \sqrt{2}x + (y - x)(1 + \sqrt{2}) = 1 + y(1 + \sqrt{2}). \end{aligned}$$

Since $|pq| = \sqrt{x^2 + y^2}$, the corresponding stretch factor is bounded in terms of x, y as follows

$$\delta(p, q) \leq \gamma(x, y) := \frac{1 + y(1 + \sqrt{2})}{\sqrt{x^2 + y^2}}. \quad (1)$$

We now consider the case $x = 1$ separately. Let $\lambda = \frac{1}{y}$, where $y = 1, 2, 3, 4, 5, \dots$, and so $\lambda = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \in (0, 1)$. According to (1) we have

$$\gamma(1, y) \leq \frac{1 + y(1 + \sqrt{2})}{\sqrt{y^2 + 1}} = \frac{\lambda + 1 + \sqrt{2}}{\sqrt{\lambda^2 + 1}} =: f(\lambda).$$

The derivative f' vanishes at $\lambda_0 = \frac{1}{\sqrt{2}+1} = \sqrt{2} - 1 = 0.4142 \dots$. On the interval $(0, 1)$: f is increasing on the interval $(0, \lambda_0)$ and decreasing on the interval $(\lambda_0, 1)$; it attains a unique maximum at $\lambda = \lambda_0$. Since $\lambda_0 \in (\frac{1}{3}, \frac{1}{2})$, we have

$$f(\lambda) \leq \max\left(f\left(\frac{1}{3}\right), f\left(\frac{1}{2}\right)\right) = f\left(\frac{1}{3}\right) = f\left(\frac{1}{2}\right) = \delta_0.$$

It remains to consider the case $x \geq 2$; according to (1) we have

$$\begin{aligned} \delta(p, q) &\leq \frac{1 + y(1 + \sqrt{2})}{\sqrt{x^2 + y^2}} \leq \frac{1 + y(1 + \sqrt{2})}{\sqrt{4 + y^2}} \\ &= \frac{(1 + \sqrt{2})(y/2) + 1/2}{\sqrt{(y/2)^2 + 1}} \leq \sqrt{(1 + \sqrt{2})^2 + 1/4} = 2.4654 \dots < \delta_0, \end{aligned}$$

where the last inequality follows from Fact 1 by setting $\lambda = y/2$.

This completes the case analysis. Observe that the above analysis is tight since there are point pairs with $x = 1, y = 2$ having pairwise stretch factor δ_0 . We have thus shown that for any $p, q \in \Lambda$, we have $\delta(p, q) \leq (3 + 2\sqrt{2})5^{-1/2}$, completing the proof of the upper bound, and thereby the proof of Theorem 1. \square

Another degree 3 spanner with stretch factor $\delta_0 = (3 + 2\sqrt{2})5^{-1/2}$. The graph G is illustrated in Fig. 4 (left). For any two lattice points $p, q \in \Lambda$, we construct a path in G . Set $p = (0, 0)$ as the origin and consider the four quadrants $W_i, i = 1, \dots, 4$, labeled counterclockwise in the standard fashion; see Fig. 4 (right). Points on the dividing lines are assigned arbitrarily to any of the two adjacent quadrants. By the symmetry of G , we can assume that q lies in one of the first two quadrants.

Case 1: $q \in W_1$. By the symmetry of G , we may assume in the analysis that $q = (x, y)$, where $0 \leq y \leq x$. Consider the path from p to q via (y, y) , that visits every lattice point on this diagonal segment as shown in Fig. 5 and let $\ell(x, y)$ denote its length.

If $y = 0$, or $x = y$, it is easily checked that the stretch factor is at most $1 + \sqrt{2}$. Assume subsequently that $x \geq y + 1$ and $y \geq 1$. A path of length $(2 + 2\sqrt{2})\lfloor y/2 \rfloor + (y \bmod 1)(2 + \sqrt{2})$ suffices to reach from $p = (0, 0)$ to (y, y) , and a path of length $\lceil (x - y)/2 \rceil \sqrt{2} + (x - y)$ suffices to reach from (y, y) to $q = (x, y)$. Thus,

$$\ell(x, y) \leq (2 + 2\sqrt{2}) \left\lfloor \frac{y}{2} \right\rfloor + (y \bmod 1)(2 + \sqrt{2}) + \left\lceil \frac{x - y}{2} \right\rceil \sqrt{2} + (x - y).$$

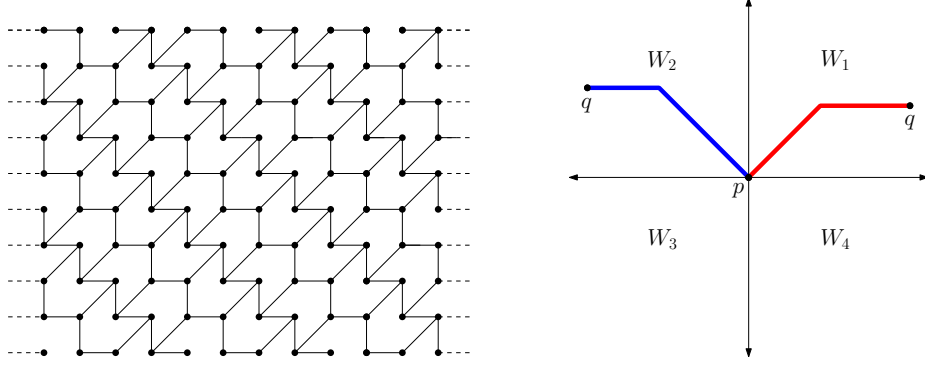


Figure 4: Left: a degree 3 spanner on Λ . Right: a schematic diagram showing the path between p, q when q lies in different quadrants of p (when $y \leq x$). The bold paths consist of segments of lengths 1 and $\sqrt{2}$.

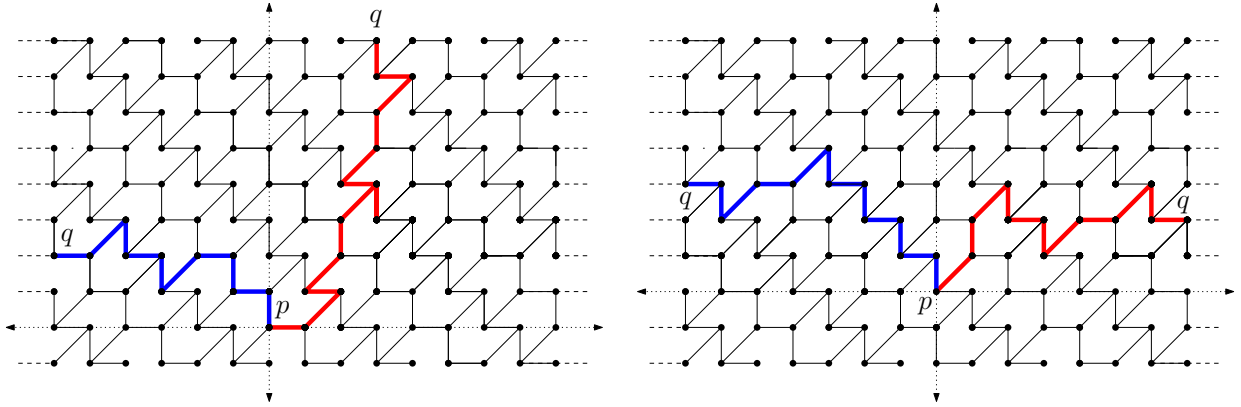


Figure 5: Illustration of various paths from p to q depending on the pattern of edges incident to p ; for $q \in W_1$ (in red) and for $q \in W_2$ (in blue). Observe that in the red path on the left, a unit vertical edge is traversed in both directions (but can be shortcut).

That is,

$$\ell(x, y) \leq \begin{cases} (2 + 2\sqrt{2})\frac{y}{2} + (x - y) + \left\lceil \frac{x - y}{2} \right\rceil \sqrt{2}, & \text{for even } y \\ (2 + 2\sqrt{2})\frac{y - 1}{2} + (2 + \sqrt{2}) + (x - y) + \left\lceil \frac{x - y}{2} \right\rceil \sqrt{2}, & \text{for odd } y. \end{cases}$$

The distance $|pq|$ equals $\sqrt{x^2 + y^2}$ in either case, and so the corresponding stretch factor (bounded in terms of x, y) is

$$\delta(p, q) \leq \gamma(x, y) := \begin{cases} \frac{\left(1 + \frac{\sqrt{2}}{2}\right)x + \frac{\sqrt{2}}{2}y + \frac{\sqrt{2}}{2}}{\sqrt{x^2 + y^2}}, & \text{for even } y \\ \frac{\left(1 + \frac{\sqrt{2}}{2}\right)x + \frac{\sqrt{2}}{2}y + \left(1 + \frac{\sqrt{2}}{2}\right)}{\sqrt{x^2 + y^2}}, & \text{for odd } y. \end{cases} \quad (2)$$

Consider first the case of even y . Since the case $y = 0$ has been dealt with, we have $y \geq 2$.

Setting $\lambda = x/y$ in (2) and using Fact 1 in the last step yields

$$\delta(p, q) \leq \gamma(x, y) = \frac{\left(1 + \frac{\sqrt{2}}{2}\right)\lambda + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2y}}{\sqrt{\lambda^2 + 1}} \leq \frac{\left(1 + \frac{\sqrt{2}}{2}\right)\lambda + \frac{3\sqrt{2}}{4}}{\sqrt{\lambda^2 + 1}} \leq \sqrt{\left(1 + \frac{\sqrt{2}}{2}\right)^2 + \frac{9}{8}} < 2.01 < \delta_0.$$

Consider now the case of odd y . We have $y \geq 1$ and $x \geq y + 1 \geq 2$. By (3) we have

$$\gamma(x, 1) = \frac{\left(1 + \frac{\sqrt{2}}{2}\right)x + (1 + \sqrt{2})}{\sqrt{x^2 + 1}} := f(x).$$

We next show that f is decreasing on the interval $[2, \infty)$. Indeed, $f'(x) = f_1(x)/(x^2 + 1)^{3/2}$, where

$$\begin{aligned} f_1(x) &= \left(1 + \frac{\sqrt{2}}{2}\right)(x^2 + 1) - x \left[\left(1 + \frac{\sqrt{2}}{2}\right)x + (1 + \sqrt{2}) \right] \\ &= \left(1 + \frac{\sqrt{2}}{2}\right) - (1 + \sqrt{2})x < 0, \text{ for } x \geq 2. \end{aligned}$$

Consequently,

$$\delta(p, q) \leq \gamma(x, 1) = f(x) \leq f(2) = \delta_0,$$

as required. Observe that the above analysis is tight for some point pairs with $x = 2, y = 1$ (that achieve stretch factor δ_0).

Consider now the remaining case $y \geq 3$. Setting $\lambda = x/y$ in (3) and using Fact 1 in the last step yields

$$\begin{aligned} \delta(p, q) \leq \gamma(x, y) &\leq \frac{\left(1 + \frac{\sqrt{2}}{2}\right)\lambda + \frac{\sqrt{2}}{2} + \left(1 + \frac{\sqrt{2}}{2}\right)\frac{1}{3}}{\sqrt{\lambda^2 + 1}} = \frac{\left(1 + \frac{\sqrt{2}}{2}\right)\lambda + \frac{1+2\sqrt{2}}{3}}{\sqrt{\lambda^2 + 1}} \\ &\leq \sqrt{\left(1 + \frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1+2\sqrt{2}}{3}\right)^2} < 2.14 < \delta_0, \end{aligned}$$

as required.

Case 2: $q \in W_2$. We may assume that $q = (-x, y)$, where $x \geq y \geq 0$. Consider the path from p to q via $(-y, y)$, that visits every lattice point on this diagonal segment as shown in Fig. 5, and let $\ell(x, y)$ denote its length. The distance $|pq|$ equals $\sqrt{x^2 + y^2}$.

If $y = 0$, it is easily checked that the stretch factor is at most $\sqrt{2}$, and so we assume subsequently that $y \geq 1$. The path length $\ell(x, y)$ is bounded from above as

$$\begin{aligned} \ell(x, y) &\leq 2y + (x - y) + \left\lceil \frac{x - y}{2} \right\rceil \sqrt{2} \leq 2y + (x - y) + \frac{x - y + 1}{2} \sqrt{2} \\ &= \left(1 + \frac{\sqrt{2}}{2}\right)x + \left(1 - \frac{\sqrt{2}}{2}\right)y + \frac{\sqrt{2}}{2} \leq \left(1 + \frac{\sqrt{2}}{2}\right)x + y. \end{aligned}$$

Setting $\lambda = x/y$ and using Fact 1 in the last step yields that the stretch factor is bounded as

$$\delta(p, q) \leq \gamma(x, y) := \frac{\left(1 + \frac{\sqrt{2}}{2}\right)x + y}{\sqrt{x^2 + y^2}} = \frac{\left(1 + \frac{\sqrt{2}}{2}\right)\lambda + 1}{\sqrt{\lambda^2 + 1}} \leq \sqrt{\left(1 + \frac{\sqrt{2}}{2}\right)^2 + 1} < \sqrt{4} = 2 < \delta_0,$$

as required.

Next, we determine the degree 4 dilation of the square lattice.

Theorem 2. *Let Λ be the infinite square lattice. Then $\delta_0(\Lambda, 4) = \sqrt{2}$.*

Proof. Trivially, the (unrestricted degree) dilation of four points placed at the four corners of a square is $\sqrt{2}$. Thus, $\delta_0(\Lambda) \geq \sqrt{2}$. To prove the upper bound, construct a 4-regular graph G on Λ by connecting every $(i, j) \in \Lambda$ with its four neighbors $(i + 1, j), (i, j + 1), (i - 1, j), (i, j - 1)$. For any two points $p, q \in \Lambda$, the Manhattan path connecting them yields a stretch factor of the form

$$\frac{x + y}{\sqrt{x^2 + y^2}} \leq \sqrt{2}, \text{ where } x, y \in \mathbf{N},$$

as required. □

Remark. It can be checked that the upper and lower bounds in Theorem 2 hold for every degree $k \geq 4$. Thus, $\delta_0(\Lambda, k) = \sqrt{2}$ for $k \geq 4$.

4 The hexagonal lattice

This section is devoted to the degree 3 and 4 dilation of the hexagonal lattice. In [16], we showed that the degree 3 dilation of the infinite hexagonal lattice is between 2 and 3. Here we establish that the exact value is $1 + \sqrt{3}$.

Theorem 3. *Let Λ be the infinite hexagonal lattice. Then $\delta_0(\Lambda, 3) = 1 + \sqrt{3}$.*

Proof. Lower bound. Consider a section of the lattice as shown in Fig. 6 (left). First, we will

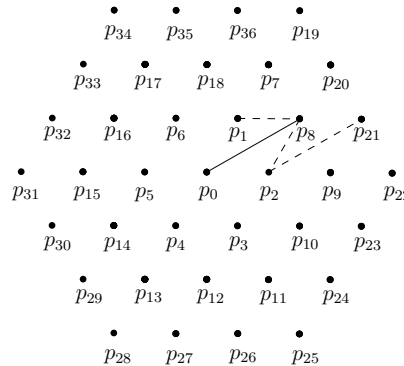


Figure 6: If an edge of length $\sqrt{3}$ is present, then the stretch factor of any plane degree 3 graph is $\geq 1 + \sqrt{3}$.

show that if an edge of length at least $\sqrt{3}$ is present, the stretch factor of any resulting plane degree 3 graph is at least $1 + \sqrt{3}$. Now assume, as we may, that the edge p_0p_8 of length $\sqrt{3}$ is present. Now consider the point pair p_1, p_2 . Clearly, $|p_1p_2| = 1$. It is easy to check that between p_1, p_2 , there are two shortest detours each of length 2, viz. $\rho(p_1, p_8, p_2)$ and $\rho(p_1, p_0, p_2)$. The next largest detours $\rho(p_1, p_8, p_9, p_2)$ and $\rho(p_1, p_0, p_3, p_2)$ have length 3 each, in which cases, $\delta(p_1, p_2) \geq 3$. Hence, without loss of any generality, consider $\rho(p_1, p_8, p_2)$, and assume that the edges p_1p_8 and p_2p_8 are present. Then,

$$\delta(p_8, p_{21}) \geq \frac{|\rho(p_8, p_2, p_{21})|}{|p_8p_{21}|} \geq 1 + \sqrt{3}.$$

A similar argument can be made for any edge e of length greater than $\sqrt{3}$, since one can always locate two lattice points lying in opposite sides of e ; as required in the above analysis. In the

remaining part of the proof, assume that no edge of length $\sqrt{3}$ or more is present. In particular, we will only consider unit length edges in our proof. Note that if every point in Λ has degree 1 in the graph, we have a matching on Λ , and hence the graph is disconnected. Thus, let p_0 be any point in Λ with degree at least 2. We have the following two¹ cases:

Case 1: $\deg(p_0) = 2$. There are 3 non-symmetric sub-cases as follows.

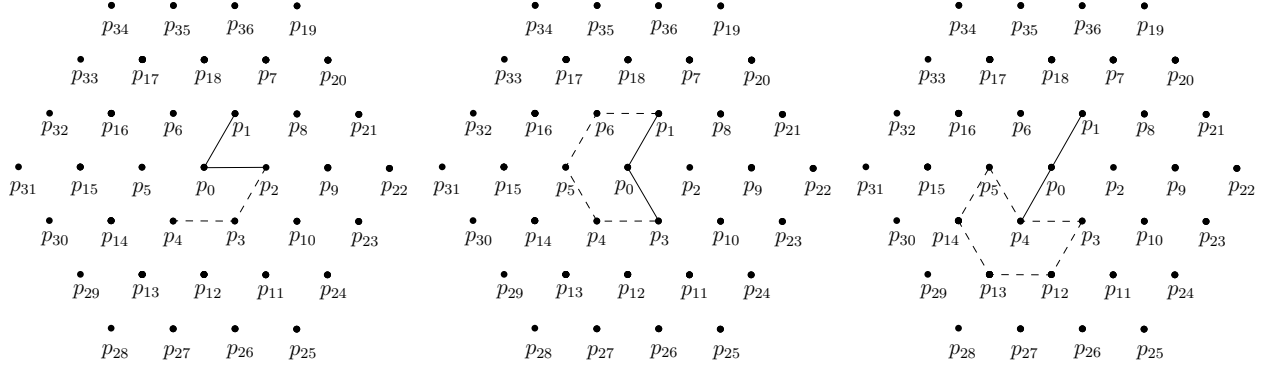


Figure 7: Illustration of *Case 1* from the proof of lower bound in Theorem 3. Left: *Case 1.1*, Middle: *Case 1.2*, Right: *Case 1.3*.

Case 1.1: Refer to Fig. 7 (left). Let the edges p_0p_1, p_0p_2 be present. Then,

$$\delta(p_0, p_4) \geq \frac{|\rho(p_0, p_2, p_3, p_4)|}{|p_0p_4|} \geq 3.$$

Case 1.2: Refer to Fig. 7 (middle). Now, let the edges p_0p_1, p_0p_3 be present. Then,

$$\delta(p_0, p_5) \geq \frac{|\rho(p_0, p_3, p_4, p_5)|}{|p_0p_5|} = \frac{|\rho(p_0, p_1, p_6, p_5)|}{|p_0p_5|} \geq 3.$$

Case 1.3: Refer to Fig. 7 (right). Let the edges p_0p_1, p_0p_4 be present. Note that if the edge p_3p_4 is absent, $\delta(p_0, p_3) \geq 3$. So, assume that p_3p_4 is present. Similarly let p_4p_5 be present otherwise $\delta(p_0, p_5) \geq 3$. Then, arguing the same way as in *Case 1.2*, $\delta(p_4, p_{13}) \geq 3$.

Case 2: $\deg(p_0) = 3$. There are 3 non-symmetric sub-cases as follows.

Case 2.1: Refer to Fig. 8 (left). Let the edges p_0p_1, p_0p_2, p_0p_3 be present. Then, by a similar argument as in *Case 1.2*, $\delta(p_0, p_5) \geq 3$.

Case 2.2: Refer to Fig. 8 (middle). Now, let the edges p_0p_3, p_0p_4, p_0p_6 be present. Clearly, if p_1p_6 is absent, $\delta(p_0, p_1) \geq 3$. Thus, assume that p_1p_6 is present. Now consider the pair p_5, p_6 . If p_5p_6 is present, then $\delta(p_6, p_{17}) \geq 3$, arguing in a similar way to *Case 1.2*. Thus, assume that p_5p_6 is absent. The shortest detour between p_5, p_6 is $\rho(p_5, p_{16}, p_6)$ which has length 2. The next largest detour has length 3. So, let the edges p_6p_{16} and p_5p_{16} be present. Now consider the pair p_{16}, p_{17} . If $p_{16}p_{17}$ is present, $\delta(p_{16}, p_{32}) \geq 3$ (analysis is similar to *Case 1.2*), otherwise, $\delta(p_6, p_{17}) \geq 3$.

Case 2.3: Refer to Fig. 8 (right). Let p_0p_2, p_0p_4, p_0p_6 be present. To achieve $\delta(p_0, p_1) = 2$, at least one of p_1p_2 or p_1p_6 needs to be present (the next largest detour has length 3). Without loss of any generality, assume that p_1p_2 is present. Now consider the pair p_2, p_3 . If p_2p_3 is present, then by *Case 2.1*, $\delta(p_2, p_9) \geq 3$. So, assume that p_2p_3 is absent. The minimum length detour is

¹As mentioned in Section 1, one can argue that degree 3 is needed for achieving a constant stretch factor. Thus, it is enough to analyze the case when $\deg(p_0) = 3$. Nevertheless, we include a complete argument.

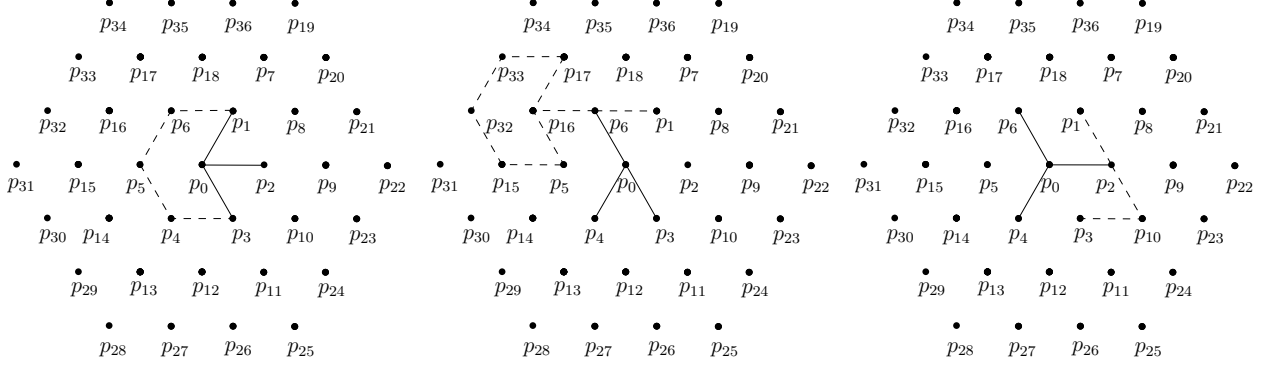


Figure 8: Illustration of *Case 2* from the proof of lower bound in Theorem 3. Left: *Case 2.1*, Middle: *Case 2.2*, Right: *Case 2.3*.

$\rho(p_2, p_{10}, p_3)$ (next largest detours have length 3 each). Thus, let p_2p_{10} and p_3p_{10} be present. Now, observe that the edges incident to p_2 form the same symmetric pattern as dealt with in *Case 2.2*, where it is shown that the stretch factor of any resulting degree 3 plane graph is at least 3.

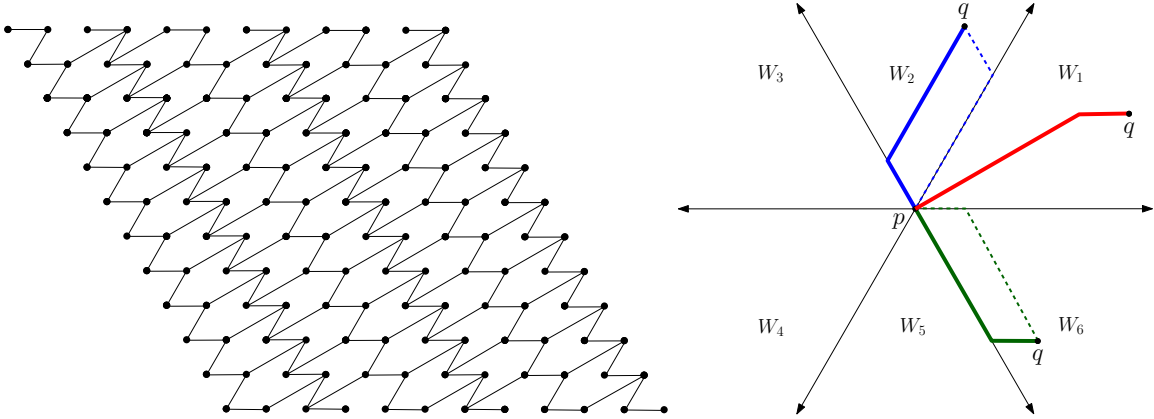


Figure 9: Left: a degree 3 plane graph on Λ . Right: a schematic diagram showing the path between p and q when q lies in different wedges determined by p . The bold paths consist of segments of lengths 1 and $\sqrt{3}$. Alternative paths are shown using dotted segments.

Upper bound. We construct a 3-regular graph G achieving $\delta_0(\Lambda, 3) \leq 1 + \sqrt{3}$, as illustrated in Fig. 9 (left). For any two lattice points $p, q \in \Lambda$, we construct a path in G . Set p as the origin, and subdivide the plane into six wedges of 60° each, centered at p , and labeled counterclockwise W_i , $i = 1, \dots, 6$, as in Fig. 9 (right). Points on the dividing lines are assigned arbitrarily to any of the two adjacent wedges. Let $\theta = \pi/3$, and consider the three unit vectors $\vec{\mu}_i = (\cos i\theta, \sin i\theta)$, for $i = 0, 1, 2$. We distinguish three cases depending on the location of q .

Case 1: $q \in W_1$ (the case $q \in W_4$ is symmetric), i.e., $\vec{q} = u\vec{\mu}_0 + v\vec{\mu}_1$, for some $u, v \in \mathbf{N}$. By the symmetry of G , we can assume that $u \geq v \geq 0$ in the analysis. Consider the path from p to q via $v\vec{\mu}_0 + v\vec{\mu}_1$ that visits every lattice point on the diagonal segment, as shown in Fig. 10 and let $\ell(u, v)$ denote its length. Observe that connecting $a\vec{\mu}_0 + a\vec{\mu}_1$ to $(a+2)\vec{\mu}_0 + (a+2)\vec{\mu}_1$ requires a length of $2 + 2\sqrt{3}$. Thus,

$$\ell(u, v) \leq (2 + 2\sqrt{3}) \left\lfloor \frac{v}{2} \right\rfloor + (2 + \sqrt{3})(v \bmod 1) + (u - v) + \left\lceil \frac{u - v}{2} \right\rceil \sqrt{3}.$$

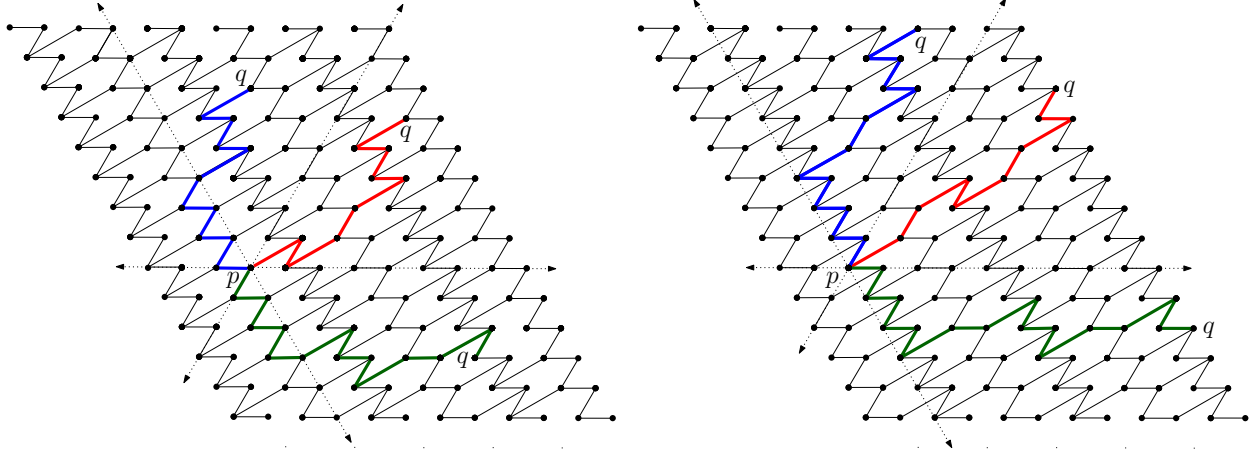


Figure 10: Illustration of various paths from p to q depending on the pattern of edges incident to p ; for $q \in W_1$ (in red), for $q \in W_2$ (in blue), and for $q \in W_6$ (in green).

For even v , $\ell(u, v)$ is bounded from above by

$$\ell(u, v) \leq \left(2 + 2\sqrt{3}\right) \frac{v}{2} + \left(1 + \frac{\sqrt{3}}{2}\right) (u - v) + \frac{\sqrt{3}}{2} = \left(1 + \frac{\sqrt{3}}{2}\right) u + \frac{\sqrt{3}}{2} v + \frac{\sqrt{3}}{2}.$$

For odd v , $\ell(u, v)$ is bounded from above by

$$\begin{aligned} \ell(u, v) &\leq \left(2 + 2\sqrt{3}\right) \frac{v-1}{2} + (2 + \sqrt{3}) + \left(1 + \frac{\sqrt{3}}{2}\right) (u - v) + \frac{\sqrt{3}}{2} \\ &= \left(1 + \frac{\sqrt{3}}{2}\right) u + \frac{\sqrt{3}}{2} v + \left(1 + \frac{\sqrt{3}}{2}\right). \end{aligned}$$

The distance $|pq|$ equals

$$\sqrt{u^2 + v^2 - 2uv \cos \frac{2\pi}{3}} = \sqrt{u^2 + v^2 + uv},$$

and so the corresponding stretch factor $\delta(p, q)$ is bounded by a function $\gamma(u, v)$ as follows

$$\delta(p, q) \leq \gamma(u, v) := \begin{cases} \frac{\left(1 + \frac{\sqrt{3}}{2}\right) u + \frac{\sqrt{3}}{2} v + \frac{\sqrt{3}}{2}}{\sqrt{u^2 + v^2 + uv}}, & \text{for even } v \\ \frac{\left(1 + \frac{\sqrt{3}}{2}\right) u + \frac{\sqrt{3}}{2} v + \left(1 + \frac{\sqrt{3}}{2}\right)}{\sqrt{u^2 + v^2 + uv}}, & \text{for odd } v. \end{cases}$$

Consider first the case of even v . We have $u \geq v \geq 0$ and $u \geq 1$ (since $u = v = 0$ is not a valid choice). We next show that $\gamma(u, v)$ is a decreasing function of v for $v \geq 0$. Indeed,

$$\frac{\partial \gamma(u, v)}{\partial v} = f(u, v) / [2(u^2 + v^2 + uv)^{3/2}],$$

where

$$\begin{aligned} f(u, v) &= \sqrt{3}(u^2 + v^2 + uv) - (2v + u) \left[\left(1 + \frac{\sqrt{3}}{2}\right)u + \frac{\sqrt{3}}{2}v + \frac{\sqrt{3}}{2} \right] \\ &= \left(\frac{\sqrt{3}}{2} - 1\right)u^2 - \left(2 + \frac{\sqrt{3}}{2}\right)uv - \frac{\sqrt{3}}{2}u - \sqrt{3}v < 0. \end{aligned}$$

Consequently,

$$\delta(p, q) \leq \gamma(u, v) \leq \gamma(u, 0) = \frac{\left(1 + \frac{\sqrt{3}}{2}\right)u + \frac{\sqrt{3}}{2}}{u} = 1 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2u} \leq 1 + \sqrt{3},$$

as required.

Consider now the case of odd v . We have $u \geq v \geq 1$. Since the expressions of $\gamma(u, v)$ for odd and even v differ by $(u^2 + v^2 + uv)^{-1/2}$, which is also a decreasing function of v , it follows that $\gamma(u, v)$ for odd v is decreasing on the same interval, in particular on the interval $v \geq 1$. Consequently,

$$\delta(p, q) \leq \gamma(u, v) \leq \gamma(u, 1) = \frac{\left(1 + \frac{\sqrt{3}}{2}\right)u + (1 + \sqrt{3})}{\sqrt{u^2 + u + 1}} \leq \frac{3}{2} + \frac{2}{\sqrt{3}} < 1 + \sqrt{3},$$

as required. To check this last inequality, let

$$h(u) = \frac{\left(1 + \frac{\sqrt{3}}{2}\right)u + (1 + \sqrt{3})}{\sqrt{u^2 + u + 1}},$$

and notice that this function is decreasing for $u \geq 1$, thus $h(u) \leq h(1) = \frac{3}{2} + \frac{2}{\sqrt{3}}$.

Case 2: $q \in W_2$ (the case $q \in W_5$ is symmetric), i.e., $\vec{q} = u\vec{\mu}_2 + v\vec{\mu}_1$, for some $u, v \in \mathbf{N}$. Consider the path from p to q via $u\vec{\mu}_2$ as shown in Fig. 10 and let $\ell(u, v)$ denote its length. (Alternatively, the path via $v\vec{\mu}_1$ can be used.) Then $\ell(u, v)$ is bounded from above as follows

$$\ell(u, v) \leq 2u + v + \left\lceil \frac{v}{2} \right\rceil \sqrt{3} \leq 2u + \left(1 + \frac{\sqrt{3}}{2}\right)v + \frac{\sqrt{3}}{2}.$$

As in Case 1, the distance $|pq|$ equals $\sqrt{u^2 + v^2 - 2uv \cos \frac{2\pi}{3}} = \sqrt{u^2 + v^2 + uv}$, and so the corresponding stretch factor is

$$\delta(p, q) \leq \gamma(u, v) := \frac{2u + \left(1 + \frac{\sqrt{3}}{2}\right)v + \frac{\sqrt{3}}{2}}{\sqrt{u^2 + v^2 + uv}}.$$

We can assume that $u \geq 1$ and $v \geq 1$ (else the stretch factor is at most $1 + \sqrt{3}$). Further, since the coefficient of u is larger than that of v in the numerator, we can assume that $u \geq v \geq 1$ when maximizing $\gamma(u, v)$. Set now $\lambda = \frac{u}{v} \geq 1$. We have

$$\gamma(u, v) = \frac{2\lambda + \left(1 + \frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2v}}{\sqrt{\lambda^2 + \lambda + 1}} \leq \frac{2\lambda + (1 + \sqrt{3})}{\sqrt{\lambda^2 + \lambda + 1}} := f(\lambda).$$

It is easy to check that $f(\lambda)$ is decreasing for $\lambda \geq 1$, hence for $u, v \geq 1$ we also have

$$\delta(p, q) \leq \gamma(u, v) \leq f(1) = 1 + \sqrt{3},$$

as required.

Case 3: $q \in W_6$ (the case $q \in W_3$ is symmetric), i.e., $\vec{q} = -u\vec{\mu}_2 + v\vec{\mu}_0$, for some $u, v \in \mathbf{N}$. By the symmetry of G , this case is symmetric to Case 2.

This completes the case analysis and thereby the proof of the upper bound. \square

Remark. In [15] we have shown that a certain 13-point section of the hexagonal lattice with six boundary points removed has degree 3 dilation at least $1 + \sqrt{3}$. It is worth noting that this subset cannot be used however to deduce that the degree 3 dilation of the hexagonal lattice is at least $1 + \sqrt{3}$. Indeed, the reason is that the absence of the respective boundary points has been explicitly invoked in that argument. This is the reason of why in the proof of the lower bound in Theorem 3 we have used a different argument.

Next we determine the degree 4 dilation of the infinite hexagonal lattice.

Theorem 4. *Let Λ be the infinite hexagonal lattice. Then $\delta_0(\Lambda, 4) = 2$.*

Proof. We first prove the lower bound. Let p_0 be any point in Λ with its six closest neighbors, say, p_1, \dots, p_6 , where $|p_0 p_i| = 1$, for $i = 1, \dots, 6$. Since $\deg(p_0) \leq 4$ in any plane degree 4 geometric spanner on Λ , there exists $i \in \{1, \dots, 6\}$ such that the edge $p_0 p_i$ is absent; we may assume that $i = 1$. Then

$$\delta(p_0, p_1) \geq \frac{|\rho(p_0, p_i, p_1)|}{|p_0 p_1|} \geq 2, \text{ where } i \in \{2, 6\}.$$

To prove the upper bound, consider the 4-regular graph G shown in Fig. 11; it remains to show that $\delta(G) \leq 2$. For any two lattice points $p, q \in \Lambda$, we construct a path in G . Consider the setup from the proof of Theorem 3. Set the lower point p as the origin $(0, 0)$. Let $\theta = \pi/3$, and consider

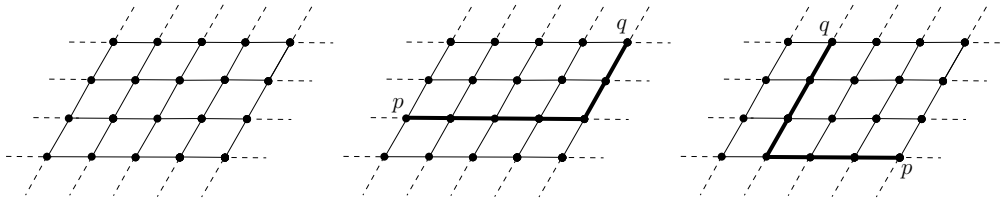


Figure 11: Left: a degree 4 plane graph G on Λ . Middle, Right: illustration of various paths from p to q depending on their relative position in Λ .

the two unit vectors $\vec{\mu}_i = (\cos i\theta, \sin i\theta)$, for $i = 0, 1$. Then $\vec{q} = \pm u\vec{\mu}_0 + v\vec{\mu}_1$, for some $u, v \in \mathbf{N}$. Since the two points can be connected by a path in G of length $u + v$, and the distance between the points is $\sqrt{u^2 + v^2} \pm uv$ (depending on their relative position in Λ), the corresponding stretch factor satisfies

$$\delta(p, q) \leq \gamma(u, v) := \frac{u + v}{\sqrt{u^2 + v^2} \pm uv} \leq 2, \quad (4)$$

as required. Indeed, the above inequalities are equivalent to $(u \pm v)^2 \geq 0$, which are obvious. \square

Remarks. 1. Another degree 4 spanner for the hexagonal lattice with stretch factor 2 appears in Fig. 12; the proof for the stretch factor is left to the reader.

2. Note that $\delta_0(\Lambda, 5) = 2$ since the bounds in Theorem 4 also hold for degree 5. Let now $k = 6$. Connecting each lattice point with all its six closest neighbors yields a planar graph with stretch

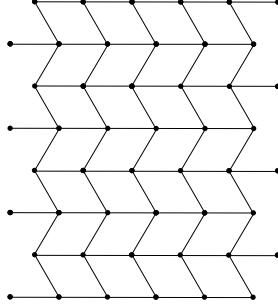


Figure 12: A degree 4 spanner on Λ with stretch factor 2.

factor $2/\sqrt{3}$. Indeed, as in (4), the stretch factor satisfies

$$\delta(p, q) \leq \gamma(u, v) := \frac{u + v}{\sqrt{u^2 + v^2 + uv}} \leq \frac{2}{\sqrt{3}},$$

where the above inequality is equivalent to $(u - v)^2 \geq 0$, which is obvious. Hence $\delta_0(\Lambda, 6) \leq 2/\sqrt{3}$. On the other hand, an argument similar to that in the proof of the inequality $\delta_0(\Lambda, 3) \geq 1 + \sqrt{3}$ shows that the presence of any edge longer than 1 would force the stretch factor to be at least 2. We may thus assume that the spanner G contains all unit edges (since no two cross each other); now the length of a shortest path in G connecting any pair of lattice points at distance $\sqrt{3}$ is 2, thus the stretch factor $2/\sqrt{3}$ is also needed. Consequently, $\delta_0(\Lambda, 6) = 2/\sqrt{3}$. It can be easily checked that $\delta_0(\Lambda, k) = \delta_0(\Lambda, 6) = 2/\sqrt{3}$ for every $k \geq 6$.

5 Concluding remarks

We have given constructive upper bounds and derived close lower bounds on the degree 3 dilation of the infinite square lattice in the domain of plane geometric spanners. We have also derived exact values for the degree 4 dilation of the square lattice along with the degree 3 and 4 dilation of the infinite hexagonal lattice. It is easy to verify that our bounds also apply for finite sections of these lattices; see [16] for some examples.

It may be worth pointing out that in addition to the low stretch factors achieved, the constructed spanners in this paper also have low weight and low geometric dilation²; see for instance [14, 17] for basic terms. That is, each of these two parameters is at most a small constant factor times the optimal one attainable.

As shown in Theorem 1, the degree 3 dilation of the infinite square lattice is at most $(3 + 2\sqrt{2})5^{-1/2}$. It would be interesting to know whether this upper bound can be improved, and so we put forward the following.

Conjecture 1. *Let Λ be the infinite square lattice. Then $\delta_0(\Lambda, 3) = (3 + 2\sqrt{2})5^{-1/2} = 2.6065\dots$*

A lighter degree 3 spanner. The graph G is illustrated in Fig. 13. It is easy to check that it is “shorter” than each of the two previous spanners of degree 3 for the square lattice analyzed in Section 3.

²When the stretch factor (or dilation) is measured over all pairs of points on edges or vertices of a plane graph G (rather than only over pairs of vertices) one arrives at the concept of *geometric dilation* of G .

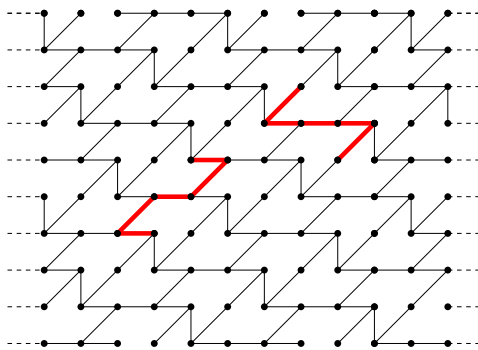


Figure 13: A lighter degree 3 spanner on the infinite square lattice. The shortest paths between point pairs with pairwise stretch factor δ_0 are shown in red.

Indeed, the average cost (length) per vertex is in this case smaller (note that some vertices have degree 2):

$$\frac{4}{5} \left(\frac{1}{2} + \frac{1}{2} + \frac{\sqrt{2}}{2} \right) + \frac{1}{5} \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) = \frac{4}{5} + \frac{3}{5} \sqrt{2} = 1.6485 \dots,$$

while that for the previous spanners it is $\frac{1}{2} + \frac{1}{2} + \frac{\sqrt{2}}{2} = 1.7071 \dots$. In particular, the total length of a square lattice section with n points is $1.6485 \dots n + o(n)$ rather than $1.7071 \dots n + o(n)$. If the stretch factor of G would also be $\delta_0 = (3 + 2\sqrt{2})5^{-1/2}$, G would be superior from the length perspective to the two spanners described in Section 3. We conjecture that the stretch factor of this lighter degree 3 spanner shown in Fig. 13 equals δ_0 .

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