Nonconvex Cases for Carpenter's Rulers

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Abstract

We consider the carpenter's ruler folding problem in the plane, i.e., finding a minimum area shape with diameter 1 that accommodates foldings of any ruler whose longest link has length 1. An upper bound of $\pi/3 - \sqrt{3}/4 = 0.614\ldots$ and a lower bound of $\sqrt{10+2\sqrt{5}}/8 = 0.475\ldots$ are known for convex cases. We generalize the problem to simple nonconvex cases: in this setting we improve the upper bound to 0.583 and establish the first lower bound of 0.073. A variation is to consider rulers with at most k links. The current best convex upper bounds are (about) 0.486 for $k = 3, 4$ and $\pi/6 = 0.523\ldots$ for $k = 5, 6$. These bounds also apply to nonconvex cases. We derive a better nonconvex upper bound of 0.296 for $k = 3, 4$.

Keywords: Carpenter's ruler, universal case, folding algorithm.

1 Introduction

Acquiring cases for rulers that are compact and easy to carry around has been a constant interest for carpenters all along. A carpenter's ruler L of n links is a chain of n line segments with endpoints $p_0, p_1, ..., p_n$, with consecutive segments connected by hinges. For $0 \le i \le n-1$, the segment $p_i p_{i+1}$ is a *link* of the ruler. A ruler with its longest link having length 1 is called a *unit ruler*. A planar folding of a ruler L is represented by the $n-1$ angles $\angle p_i p_{i+1} p_{i+2} \in [0, \pi]$, for $0 \le i \le n-2$. A *case* is a planar shape whose boundary is a *simple* closed curve (i.e., with no self-intersections); in particular, a case has no interior holes.

Obviously a unit ruler requires a case whose diameter is at least one; on the other hand, there exist cases of unit diameter that allow folding of *any* unit ruler inside, e.g., a disk of unit diameter, regardless of the number of links in the ruler. A ruler L can be folded inside a case S if and only if there exists a point $p \in S$ and a folding of L such that all the points on L are in S when p_0 is placed at p. In a folded position of the ruler, its links may cross each other; an example is shown in Figure 1 (right).

A case is said to be *universal* if any unit ruler (or equivalently, all unit rulers) can be folded inside it. Obviously, the diameter of any universal case is at least 1. Călinescu and Dumitrescu[4] asked for the minimum area of a convex universal case of unit diameter; see also [2, Problem 9, p. 461]. A disk of unit diameter and the Reuleaux triangle with one arc removed (called $R2$), were shown to be universal by the authors [4]. $R2$ is depicted in Figure 1, its area is $\frac{\pi}{3} - \frac{\sqrt{3}}{4} = 0.614\ldots$; it is the current best upper bound for the area of a convex universal case. For any n-link unit ruler $p_0, p_1, ..., p_n$, a folding of it inside $R2$ such that all p_i's lie on the circular arcs can be computed in $O(n)$ time. The authors [4] also achieved a lower bound of 0.375 using 3-link rulers, and this was further improved by Klein and Lenz [9] to 0.475 using 5-link rulers.

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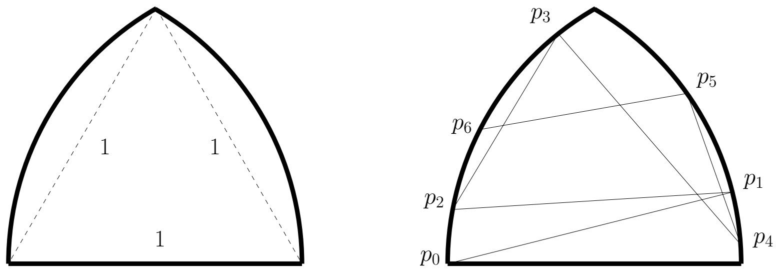

Figure 1: Left: convex universal case $R2$ (in bold lines). Right: folding a 6-link unit ruler $p_0p_1p_2p_3p_4p_5p_6$ into $R2$.

A case is *k-universal* if any unit ruler with at most k links can be folded into it. In the problem of finding a universal case with minimum area, the number of links (or the total length) of the rulers is irrelevant. However, it is a natural to ask if fewer links in the rulers allow better bounds. Alt et al. [1] studied convex universal cases for rulers with a small number of links, for which frequently better upper bounds can be achieved.

Since a universal case has unit diameter, it must be contained in a *lens* of radius 1, namely the intersection of two disks of unit radius passing through the centers of each other; see Figure 2. Interestingly, it was shown by Klein and Lenz [9] that no subset of $R2$ with a smaller area is universal. All previous work has focused on convex cases; the lower bounds were derived by minimizing the areas of the convex hulls of specific rulers used in the respective arguments.

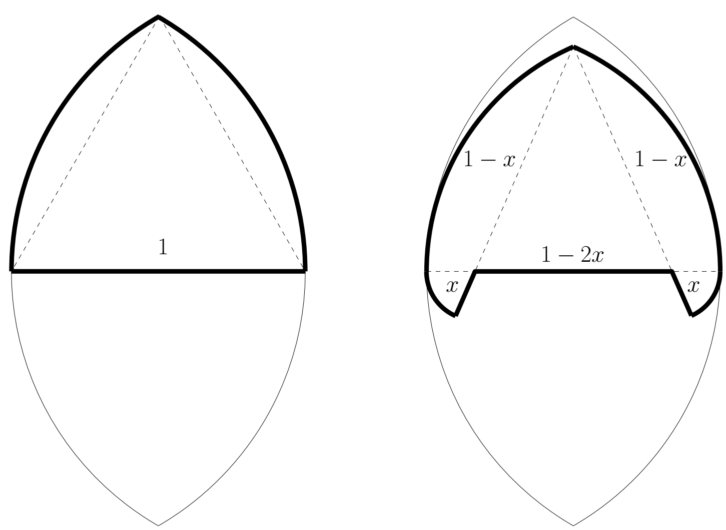

Figure 2: Universal cases (in bold lines) are contained in a lens of radius 1. Left: convex universal case $R2$. Right: nonconvex universal case C for some $x \in [0, 0.5]$.

Călinescu and Dumitrescu [4] also asked whether the convexity of the case makes any difference. Here we give a first partial answer to this question. Our main result concerning nonconvex universal cases is summarized in the following.

Theorem 1. *There exists a nonconvex universal case C of unit diameter and area at most* 0.583. *The folding of any unit ruler with n links inside C can be computed in $O(n)$ time. On the other hand, the area of any simple nonconvex universal case of unit diameter is at least* 0.073.

For the problem of finding k-universal cases, our main result for $k = 4$ is summarized in the following.

Theorem 2. *There exists a nonconvex* 4*-universal case* $C2$ *of unit diameter and area at most* 0.296*. The folding of any unit ruler with at most* 4 *links inside* $C2$ *can be computed in* $O(1)$ *time.*

Table 1 and Table 2 summarize the known and new bounds for convex and arbitrary (convex and nonconvex) cases, respectively.

Convex	Universal	3-universal	4-universal	5-universal	6-universal
Upper bounds	0.615	0.486	0.486	0.524	0.524
Lower bounds	0.475	0.375	0.375	0.475	0.475

Table 1: Known bounds for convex cases.

Arbitrary	Universal	3-universal	4-universal	5-universal	6-universal
Upper bounds	**0.583**	**0.296**	**0.296**	0.524	0.524
Lower bounds	**0.073**	**0.038**	**0.038**	**0.073**	**0.073**

Table 2: Known and new (in bold) bounds for arbitrary cases.

In Section 2, we prove that the nonconvex case C drawn in bold lines in Figure 3 is a universal case for any $x \in [0, \frac{1}{2}]$. Its area is at most 0.583 (achieved when $x = 0.165$), i.e., smaller than the area of $R2$. Notice that the case whose boundary is the convex hull of C is a convex universal case whose area is about 0.694, i.e., larger than the area of $R2$.

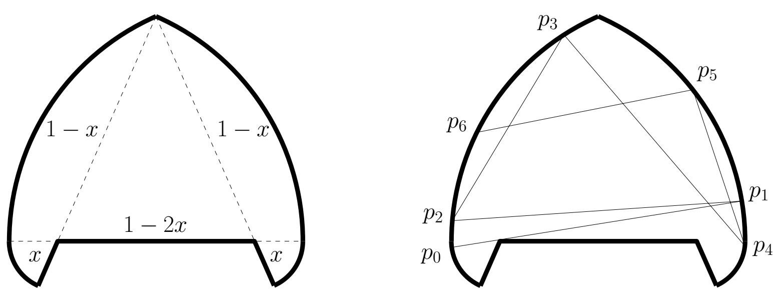

Figure 3: Left: nonconvex universal case C for some $x \in [0, 0.5]$; the shaded trapezoid can be discarded. Right: folding of a 6-link unit ruler $p_0p_1p_2p_3p_4p_5p_6$ into C.

In Section 3, lower bounds for nonconvex universal cases are considered; only areas required by the simplicity of the case boundary are taken into account. We first derive a lower bound of 0.038 using a suitable 3-link ruler, and then extend the calculation to a suitable 5-link ruler and improve the lower bound to 0.073.

In Section 4, the problem of finding k-universal cases for $k = 4$ is considered. We construct another nonconvex case $C2$ with unit diameter. It is proved to be 4-universal with an algorithm for folding unit rulers with at most 4 links inside it. $C2$ has area at most 0.296, smaller than the current best upper bound for the area of a convex 4-universal case, which is (about) 0.486.

Related work. If one insists on a unidimensional folding, Hopcroft et al. [8] observed that the minimum folded length can be at least $2 - \varepsilon$ for any $\varepsilon > 0$ with a suitable unit ruler. Moreover, determining the shortest interval, i.e., unidimensional case, into which a given ruler can be folded is NP-complete [8]. A 2-approximation algorithm [8] and a several polynomial-time approximation schemes [4] are available.

A *universal cover* (that we also call here *universal case*) for a family of sets is a simply connected set that contains a copy (under congruence or translation) of each set in the family. Geometric questions on finding minimum universal cases for various families of sets have been studied for over a hundred years. Several measures of minimality such as smallest area and shortest diameter have been considered. See [2, Chapter 14.1] and the references therein.

Leo Moser's worm problem is one of the most famous problems of this type. It asks for a planar set with minimum area that can be used to cover any curve of length one. Meir showed that a closed halfdisk of unit diameter is a universal case for all curves of unit length. Its area is $\pi/8 < 0.3927$. Gerriets and Poole [6] constructed a smaller universal case from a rhombus whose long diagonal is of unit length and the larger angles are 120°; see Figure 4 (left). Its area is $1/(2\sqrt{3}) < 0.2887$. Better

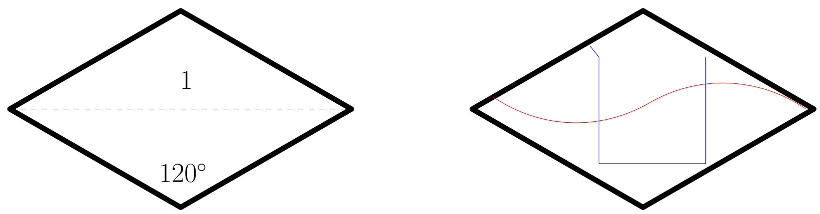

Figure 4: A universal case (in bold lines) for Moser's worm problem. Left: a rhombus with a long diagonal of unit length and vertex angles of 60° and 120°. Right: two plane curves of length 1 contained in the case.

upper bounds were gradually obtained over two decades since the 1970s; see [11, 14, 15]. The current best upper bound for convex cases is 0.274, derived by Norwood and Poole [10]. For nonconvex cases, a better upper bound of 0.246 was achieved by Hansen [7]. The current best convex lower bound of 0.2194 is due to Wetzel [15] and dates back to 1973.

One of the oldest problems concerning universal cases was posed by Lebesgue. The problem asks for a smallest area convex universal case that contains a congruent copy of every planar set of unit diameter. The union of a Reuleaux triangle with diameter one and a unit circle where a pair of the triangle vertices is a diameter of the circle makes a universal case of area $1.0046\ldots$ [5]. Pál [12] (see also [2, Problem 1, p. 457]) proved an upper bound of 0.8454 with his truncated hexagon, i.e., a regular hexagon circumscribed to a unit circle, with two corners cut off. Pál also derived a lower bound of 0.8257 which was further increased to 0.832 by Braß and Sharifi [3].

Similar problems in which translation (but no rotation) is allowed were also studied. For the same problem, a universal case under translation is also a universal case under congruence. The direct analogue of Lebesgue's problem asks for a smallest universal case that contains a translate of every planar set of unit diameter. The unit square is known to be a universal case under translation (and congruence). One of the oldest results for universal cases under translation is due to Pál [13] who proved that the smallest universal translative case for all open curves of unit length is an equilateral triangle of unit height.

2 Upper bound

In this section we prove the upper bound in Theorem 1 using the nonconvex shape C shown in Figure 5. The case $C = abcdefg$ is constructed as follows.

- $|ac| = |af| = |bg| = 1$

- $|bd| = |cd| = |ef| = |eg| = x,\ x \in [0, \frac{1}{2}]$
- The arcs ab and gf are centered at e with radii $1-x$ and x respectively
- The arcs ag and bc are centered at d with radii $1-x$ and x respectively

Notice that when $x = \frac{1}{2}$, C becomes a disk with diameter 1; and when $x = 0$, C is identical to $R2$. We show below that for any $x \in [0, \frac{1}{2}]$, C is a universal case with diameter 1. Choosing $x = 0.165$ yields a universal case with area ≤ 0.583; notice that this area is smaller than $\frac{\pi}{3} - \frac{\sqrt{3}}{4} = 0.614\ldots$, the area of $R2$, the current smallest convex universal case.

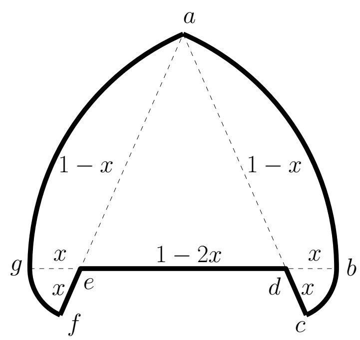

Figure 5: Nonconvex universal case C (in bold lines).

2.1 Diameter of C

We show that C has diameter 1 for any $x \in [0, \frac{1}{2}]$. The diameter is given by a pair of points on the convex hull, thus it suffices to consider points on arcs ab, bc, fg, ga and segment cf. Let p and p' be two points on the convex hull of C.

Fix p on arc ab. If p' is on arc ab, $|pp'| \leq |ab| < |ac| = 1$. If p' is on arc bc or segment cf, $|pp'| \leq |ac| = 1$. If p' is on arc fg, extend segment pe until it intersects arc fg at point p''. If $p' = p''$, $|pp'| = |pe| + |ep''| = 1$; otherwise, segments pe, ep', pp' form a triangle, thus, $|pp'| < |pe| + |ep'| = |ae| + |ef| = 1$. If p' is on arc ga, $|pp'| \leq |bg| = 1$.

Fix p on arc bc. If p' is on segment cf, $|pp'| \leq |bf| < |bg| = 1$. If p' is on arc fg, $|pp'| \leq |bg| = 1$. By symmetry, C has diameter 1.

2.2 Algorithm for folding a ruler inside C

We show that a folding of any unit ruler with n links inside C can be computed in $O(n)$ time. We adapt the algorithm introduced in [4] to work with our case C. Fix the first free endpoint at some (arbitrary) point p on a circular arc. Iteratively fix the next point of the ruler at some intersection point between the arcs of C and the circle centered at p with radius the length of the current link.

Notice that for any point p on the circular arcs of C, and for any $t \in [0, 1]$, there exists at least one point p' on these arcs such that $|pp'| = t$. This guarantees the existence of the intersection points used in the iterative steps of the above algorithm.

2.3 Minimizing the area of C

The area of C is the sum of areas of the sectors dag, dbc, eab and efg minus the area of the triangle Δade. In the triangle Δade, we have $\angle ade = \arccos \frac{1-2x}{2-2x}$. The sectors dag and eab have the same area $\frac{(1-x)^2}{2}\arccos\frac{1-2x}{2-2x}$. The sectors dbc and efg have the same area $\frac{x^2}{2}\arccos\frac{1-2x}{2-2x}$.

The triangle Δade has area $\frac{1-2x}{4}\sqrt{3-4x}$. It follows that

$$\begin{aligned}\text{area}(C) &= 2\text{area}(\text{sector } dag) + 2\text{area}(\text{sector } dbc) - \text{area}(\Delta ade)\\ &= (1-x)^2\arccos\frac{1-2x}{2-2x} + x^2\arccos\frac{1-2x}{2-2x} - \frac{1-2x}{4}\sqrt{3-4x}\\ &= (1-2x+2x^2)\arccos\frac{1-2x}{2-2x} + \frac{2x-1}{4}\sqrt{3-4x}.\end{aligned}$$

Minimizing the area function on $x \in [0,1)$ using Mathematica (see Appendix A) yields that for $x = 0.165$, we have area$(C) \le 0.583$.

3 Lower bound

In this section we prove the upper bound in Theorem 1. We start with Lemma 1 (in Subsection 3.1), which gives a lower bound of 0.038 for the area required by a suitable 3-link ruler. As it turns out, this lower bound is the best possible for all 3-link rulers. Lemma 1 will be reused in Subsection 3.2 when deriving a lower bound for 5-link rulers, improving this first bound to 0.073.

3.1 Lower bound with one 3-link ruler

For 3-link rulers, it suffices to consider the sequence of lengths $1, t, 1$ with $t \in (0,1)$. Indeed, given a folding of ruler $1, t, 1$, and an arbitrary unit 3-link ruler with links a, t, b, make the t-links of the two rulers coincide, and fold the a- and b-links over the two unit links; the resulting folding is a valid one in the same case required by the $1, t, 1$ ruler.

For the 3-link ruler with link lengths $1, t, 1$, the two 1-links must intersect otherwise the diameter constraint will be violated, see Figure 6. The shaded triangle is the only area that counts for the nonconvex lower bound.

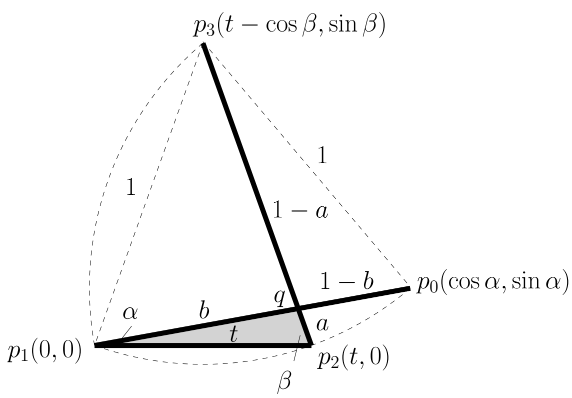

Figure 6: For a 3-link ruler $1, t, 1$, where t is fixed, the area of the shaded triangle is minimized when $|p_0p_3| = |p_1p_3| = 1$.

Lemma 1. *For any $t \in (0,1)$, the shaded triangle in Figure 6 is minimized when $\alpha = \arccos\frac{t}{2} - \frac{\pi}{3}$ and $\beta = \arccos\frac{t}{2}$.*

Proof. By symmetry, we can assume that $\alpha \le \beta$. Denote the area of the shaded triangle Δp_1p_2q by S. Since the triangle has base t, its height h determines the area. The height h is the distance between p_1p_2 and the intersection point between p_0p_1 and p_2p_3. For any fixed $\alpha \in [\arccos\frac{t}{2} - \frac{\pi}{3}, \arccos\frac{t}{2}]$, the area is minimized when β is minimized without violating the diameter constraint $|p_0p_3| \le 1$. Denote this angle by $\beta(\alpha)$; $\beta(\alpha)$ is a monotonically decreasing function that can be determined by computing the intersection of two circles of radius 1 centered at p_0 and p_2. In the following discussion, we will refer to this angle by β.

It suffices to express the area S as a function of two parameters, t and α. In fact, $h\cot\alpha + h\cot\beta = t$, so

$$S(t,\alpha) = \frac{th}{2} = \frac{t^2}{2(\cot\alpha + \cot\beta)}.$$

Taking derivative with respect to α, we have

$$\frac{dS(t,\alpha)}{d\alpha} = \frac{t^2}{2(\cot\alpha + \cot\beta)^2}\left(\frac{1}{\sin^2\alpha} + \frac{1}{\sin^2\beta}\frac{d\beta}{d\alpha}\right).$$

To see that S is minimized when α is minimized, we need to show that $\frac{dS(t,\alpha)}{d\alpha} > 0$, i.e.,

$$\frac{1}{\sin^2\alpha} + \frac{1}{\sin^2\beta}\frac{d\beta}{d\alpha} > 0, \text{ or } \frac{d\beta}{d\alpha} > -\frac{b^2}{a^2}.$$

Suppose that $p_1 = (0,0)$ and $p_2 = (t,0)$; then $p_0 = (\cos\alpha, \sin\alpha)$ and $p_3 = (t - \cos\beta, \sin\beta)$. Since $|p_0p_3| = 1$, we have

$$(t - \cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 = 1, \text{ or}$$
$$t^2 + 1 - 2t\cos\beta - 2t\cos\alpha + 2\cos(\alpha+\beta) = 0.$$

Taking derivative with respect to α, we have

$$2t\sin\beta\frac{d\beta}{d\alpha} + 2t\sin\alpha - 2\sin(\alpha+\beta)(1 + \frac{d\beta}{d\alpha}) = 0, \text{ or}$$
$$\frac{d\beta}{d\alpha} = \frac{\sin(\alpha+\beta) - t\sin\alpha}{t\sin\beta - \sin(\alpha+\beta)}.$$

Notice that in the shaded triangle Δp_1p_2q,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{t}{\sin(\alpha+\beta)}, \text{ so } \frac{d\beta}{d\alpha} = -\frac{1-a}{1-b}.$$

Thus $\frac{dS(t,\alpha)}{d\alpha} > 0$ is equivalent to

$$\frac{d\beta}{d\alpha} = -\frac{1-a}{1-b} > -\frac{b^2}{a^2}, \text{ or } b^2 - b^3 > a^2 - a^3.$$

This inequality holds if $0 < a < 2/3$ and $a < b < (1-a)/2 + \sqrt{(1+3a)(1-a)}/2$. We know this is true since in triangle Δp_0p_3q, $(1-a)+(1-b) > 1$. Recall $\alpha \le \beta$, so $a \le b$. If $a > 1/2$, $(1-a)+(1-b) \le 2(1-a) < 1$, so $a \le 1/2$. And $b < 1-a < (1-a)/2 + \sqrt{(1-a)^2}/2$.

Observe that α and the correspondingly β are determined when $|p_1p_3| = 1$; see Figure 6. Moreover, this value of α is the minimum possible; indeed, if α is getting smaller, either $|p_0p_3|$ or $|p_1p_3|$ will violate the diameter constraint. In the isosceles triangle $\Delta p_1p_2p_3$, we have $\beta = \alpha + \angle p_0p_1p_3$ and $\cos\beta = \frac{t}{2}$. In the equilateral triangle $\Delta p_0p_1p_3$, we have $\angle p_0p_1p_3 = 60°$. So $\beta = \alpha + \frac{\pi}{3} = \arccos\frac{t}{2}$. □

Now we are ready to obtain our first lower bound on simple nonconvex cases. By Lemma 1,

$$S(t,\alpha) \geq U(t) := \frac{t^2}{2(\cot(\arccos \frac{t}{2} - \frac{\pi}{3}) + \cot \arccos \frac{t}{2})}. \tag{1}$$

It is easy to check that $U(0.676) \geq 0.038$, as desired. For $t \in (0,1)$, $U(t)$ attains its maximum value for $t \approx 0.676$, and this maximum value is the best possible bound that can be obtained using a single 3-link ruler.

3.2 Lower bound with one 5-link ruler

Consider a special ruler with 5 links of lengths 1, 0.6, 1, 0.6, 1 and folding angles shown in Figure 7. Recall that all the 1-links must pairwise intersect. Since the ruler is symmetric, w.l.o.g., we can assume that $\beta \geq \gamma$. The following lemma gives a better lower bound on the area of a universal case using this ruler.

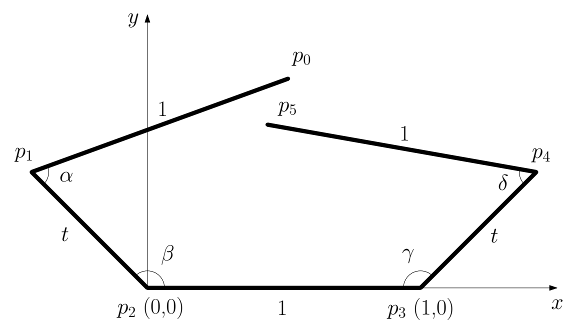

Figure 7: Legend for the 5-link ruler used (in bold lines). The figure does not show a valid folding, since e.g., the unit links do not pairwise intersect.

Lemma 2. *The minimum area of a simple (nonconvex) case of unit diameter required by folding the ruler 1, 0.6, 1, 0.6, 1 inside it is at least* 0.073*.*

Proof. Put $t = 0.6$. The Cartesian coordinate is set up as follows: fix the origin at p_2 and let the x-axis pass through p_3. We have $p_2 = (0,0)$, $p_3 = (1,0)$, $p_1 = (t\cos\beta, t\sin\beta)$ and $p_4 = (1 - t\cos\gamma, t\sin\gamma)$. Recall that the case is required to be simple, i.e., no self-intersections or holes are allowed. According to the analysis of 3-link rulers, $\beta, \gamma \in [\arccos \frac{t}{2} - \frac{\pi}{3}, \arccos \frac{t}{2}]$. We distinguish four cases according to the angles β and γ and to whether the two t-links intersect or not.

Case 1: The two t-links do not intersect. This case includes the situation that p_3p_4 is folded below p_2p_3. As shown in Figure 8 (left), each shaded triangle is minimized using Lemma 1.

$$\beta = \gamma = \arccos \frac{t}{2} = 72.54\ldots^\circ, \quad \alpha = \delta = \arccos \frac{t}{2} - \frac{\pi}{3} = 12.54\ldots^\circ.$$

While this is not a valid folding since the two 1-links p_0p_1 and p_4p_5 do not intersect, it gives a valid lower bound since for any fixed β and γ, increasing α or δ (to make these links intersect) will increase the total area. By (1), the lower bound for Case 1 is

$$2U(t) = \frac{t^2}{\cot(\arccos \frac{t}{2} - \frac{\pi}{3}) + \cot \arccos \frac{t}{2}} \geq 0.074.$$

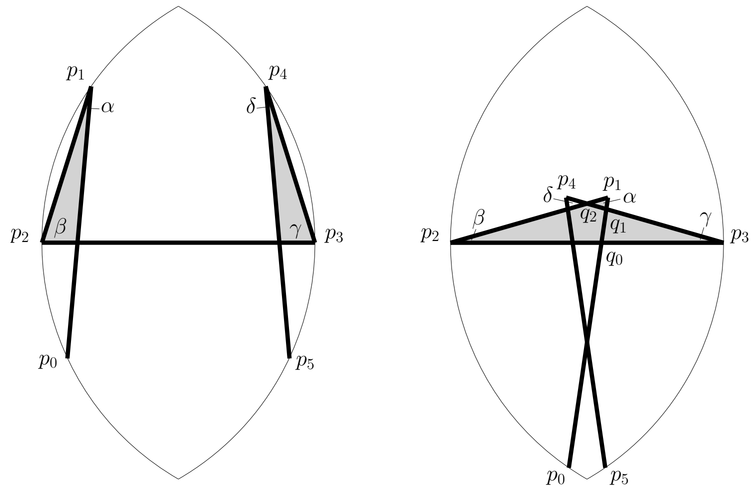

Figure 8: Case 1 (left) and Case 2 (right): the lower bounds are given by the shaded areas in each case.

Case 2: The two t-links intersect and both β and γ are at least $16°$. As shown in Figure 8 (right), increasing β or γ will enlarge the upper shaded area consisting of the triangles $\Delta q_0p_1p_2$ and $\Delta q_0p_3q_1$. The area of the triangle below p_2p_3 will decrease but we simply ignore it when computing the lower bound in this case. Similar to the case of 3-link rulers, when $\beta = \gamma = 16°$, α is to be minimized under the constraint $|p_0p_3| \leq 1$ (otherwise the area of the upper right small triangle $\Delta q_1p_1q_2$ will increase), namely the point p_0 is the intersection point below the x-axis of the circles of unit radius centered at p_3 and p_1:

$$(x-1)^2 + y^2 = 1 \quad \text{and} \quad (x - t\cos\beta)^2 + (y - t\sin\beta)^2 = 1.$$

It follows (see Appendix B) that the slope of p_0p_1 is $\tan(\alpha+\beta) = 6.9304\ldots$.

In the triangle $\Delta q_0p_1p_2$, the height equals $t\sin\beta$ and the base $b = |p_2q_0|$ is the difference between the projections of the segments p_2p_1 and q_0p_1 on the x-axis, hence

$$b = t\cos\beta - \frac{t\sin\beta}{\tan(\alpha+\beta)}. \tag{2}$$

Triangle $\Delta q_0p_3q_1$ has base $1-b$. Its height h equals to the y-coordinate of q_1 which is the intersection point of lines p_0p_1 and p_3p_4. The equations of the lines p_0p_1 and p_3p_4 are

$$y = \tan(\alpha+\beta)(x - t\cos\beta) + t\sin\beta \quad \text{and} \quad y = (1-x)\tan\gamma,$$

respectively, hence the y-coordinate of their intersection is

$$h = \frac{(t\sin\beta + (1 - t\cos\beta)\tan(\alpha+\beta))\tan\gamma}{\tan\gamma + \tan(\alpha+\beta)}. \tag{3}$$

Plugging the value of $\tan(\alpha+\beta)$ into equations (2) and (3) yields $b = 0.5528\ldots$ and $h = 0.1231\ldots$. Consequently, the total shaded area is the sum of the two areas of triangles $\Delta q_0p_1p_2$ and $\Delta q_0p_3q_1$, namely

$$\frac{bt\sin\beta + (1-b)h}{2} \geq 0.073. \tag{4}$$

Case 3: The two t-links intersect and $\beta \geq 16°, \gamma \leq 16°$. In this case, the lower bound consists

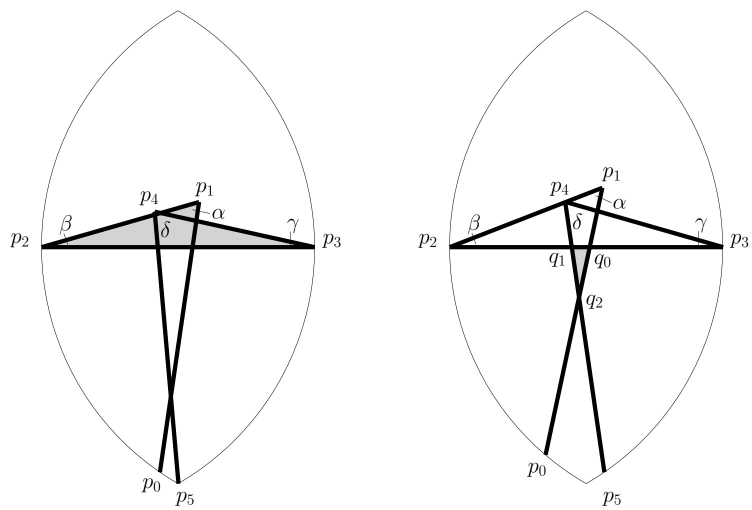

Figure 9: Case 3: area above (left) and area below (right). The lower bound is given by the sum of the two shaded areas.

of two parts, the minimum shaded areas above and below p_2p_3, denoted by S_a and S_b, respectively. As shown in Figure 9 (left), with a similar argument as in Case 2, the minimum shaded area above p_2p_3 is achieved when $\beta = 16°, \gamma = \arccos\frac{t}{2} - \frac{\pi}{3} = 12.54\ldots°$ (which is the minimum value), and α is minimized under the constraint $|p_0p_3| \leq 1$. Plugging in these values into (2), (3) and (4) in Case 2 yields $S_a \geq 0.067$.

Observe that when β and γ increase, α and δ can take smaller values under the constraints $|p_0p_3| \leq 1$, $|p_2p_5| \leq 1$ and thus form a smaller triangle below p_2p_3. So the area of triangle $\Delta q_0q_1q_2$ is minimized when both β and γ take the maximum values, i.e., $\gamma = 16°$ and $\beta = \arctan\frac{t\sin\gamma}{1-t\cos\gamma} = 21.343\ldots°$ is chosen such that p_4 lies on p_1p_2 (p_1p_2 and p_3p_4 need to intersect). Then, both α and δ are minimized under the diameter constraints. Similar to Appendix B, $\tan(\alpha+\beta) = 4.6695\ldots$ and $\tan(\gamma+\delta) = 6.9304\ldots$. This configuration is shown in Figure 9 (right). Similar to (2), we have

$$\begin{aligned} |p_2q_0| &= t\cos\beta - \frac{t\sin\beta}{\tan(\alpha+\beta)}, \\ |q_1p_3| &= t\cos\gamma - \frac{t\sin\gamma}{\tan(\gamma+\delta)}. \end{aligned} \tag{5}$$

The base of triangle $\Delta q_0q_1q_2$ is $b = |p_2q_0| + |q_1p_3| - 1$. The height h of this triangle is the absolute value of the y-coordinate of q_2, the intersection point of lines p_0p_1 and p_4p_5. The equation of line p_0p_1 is

$$y = \tan(\alpha+\beta)(x - t\cos\beta) + t\sin\beta,$$

and the equation of line p_4p_5 is

$$y = \tan(\gamma+\delta)(1 - t\cos\gamma - x) + t\sin\gamma.$$

Solving for their intersection point gives

$$h = \frac{\tan(\alpha+\beta)\tan(\gamma+\delta)(t\cos\beta + t\cos\gamma - 1)}{\tan(\alpha+\beta)+\tan(\gamma+\delta)} - \frac{t\tan(\gamma+\delta)\sin\beta + t\tan(\alpha+\beta)\sin\gamma}{\tan(\alpha+\beta)+\tan(\gamma+\delta)}. \tag{6}$$

It follows that $S_b = \frac{1}{2}hb \geq 0.006$, and consequently, the minimum total shaded area is $S_a + S_b \geq 0.073$.

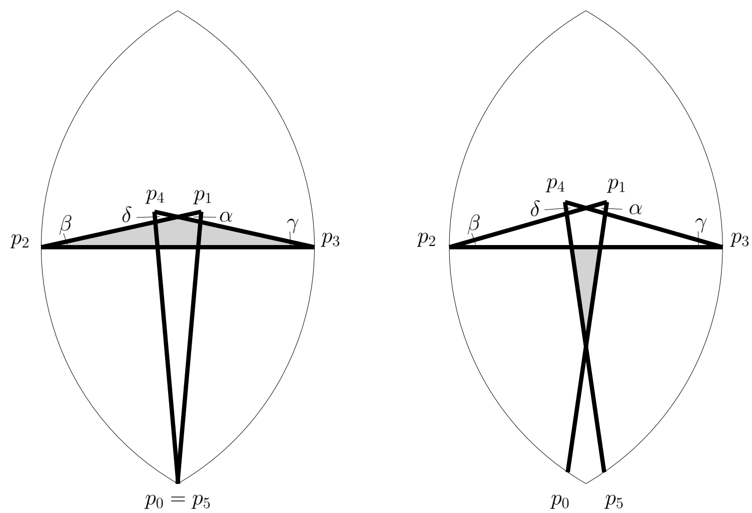

Figure 10: Case 4: area above (left) and area below (right). The lower bound is given by the sum of the two shaded areas.

Case 4: both β *and* γ *are at most* $16°$. Since $t = 0.6$, the two t-links must intersect. Similar to Case 3, the lower bound is calculated as the sum of minimized areas of shaded triangles above and below p_2p_3. For the triangle above p_2p_3, recall that β and γ both have the minimum possible value, $\arccos \frac{t}{2} - \frac{\pi}{3}$, as shown in Figure 10 (left). The minimized isosceles triangle above p_2p_3 has base 1 and height $\frac{\tan \beta}{2}$. Its area is

$$S_a = \frac{\tan(\arccos \frac{t}{2} - \frac{\pi}{3})}{4} \geq 0.055.$$

The area of the triangle below p_2p_3 is minimized when both β and γ take the maximum value, $16°$. Using (5) and (6) in Case 3 and $\alpha = \delta$, $\beta = \gamma$, the triangle below p_2p_3 has base

$$b = 2\left(t \cos \beta - \frac{t \sin \beta}{\tan(\alpha + \beta)}\right) - 1$$

and height

$$h = \frac{(2t \cos \beta - 1)\tan(\alpha + \beta)}{2} - t \sin \beta.$$

Its area is $S_b = \frac{hb}{2} \geq 0.019$. The minimum total shaded area is $S_a + S_b \geq 0.074$.

In summary, by Cases 2 and 3 of the analysis, the minimum nonconvex area required by folding the ruler 1, 0.6, 1, 0.6, 1 within a case of unit diameter is at least 0.073. □

4 k-universal cases

In this section, we consider the problem of finding k-universal cases of minimum areas. Let A be the smallest area of a convex universal case and B be the smallest area of an arbitrary (convex or nonconvex) universal case; obviously, $B \leq A$. Let A_k be the smallest area of a convex k-universal case and B_k be the smallest area of an arbitrary (convex or nonconvex) k-universal case. It is clear that $B_k \leq A_k$ for any k; moreover, we have $A_k \leq A$ and $B_k \leq B$ for any k. In addition, we have $A_i \leq A_j$ and $B_i \leq B_j$ for any $i < j$. The problem of finding better bounds for A_k was first studied by Alt et al. [1]; the authors proved (and we verified) that $A_3 \leq A_4 \leq 0.486$ and $A_5 \leq A_6 \leq 0.524$.

In Section 3 we showed that $B_3 \geq 0.038$ and $B_5 \geq 0.073$; consequently, $B \geq 0.073$. In this section we show that $B_4 \leq 0.296$. Next, we briefly review the upper bounds on A_3, A_4, A_5, A_6 mentioned previously, results on which we build later on.

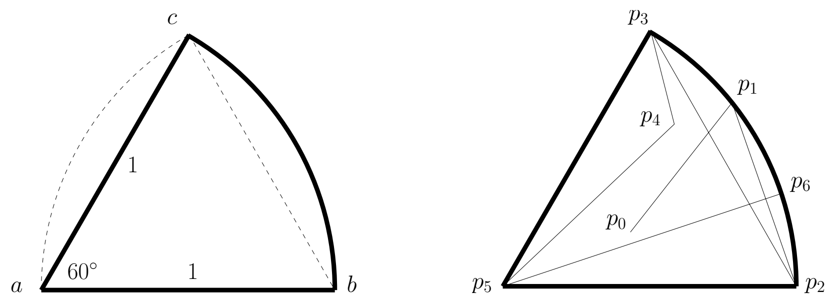

Figure 11: Left: 6-universal case $R1$ (in bold lines). Right: folding of a 6-link unit ruler $p_0p_1p_2p_3p_4p_5p_6$ into $R1$.

Replacing one of the two circular arcs in $R2$ (Figure 1) with its chord results in the case $R1$, depicted in Figure 11, a circular sector with radius one and center angle 60°; its area is $\pi/6$. Alt et al. [1] proved that $R1$ is 6-universal but not 7-universal. Thus

$$A_6 \leq \text{area}(\text{sector } abc) = \pi/6 = 0.532\ldots.$$

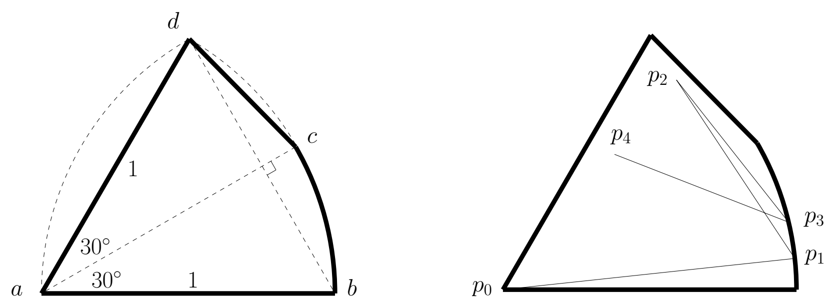

Figure 12: Left: 4-universal case $R1/2$ (in bold lines). Right: folding of a 4-link unit ruler $p_0p_1p_2p_3p_4$ into $R1/2$.

Further replacing half of the remaining circular arc in $R1$ with a line segment produces the case $R1/2$ shown in Figure 12. $R1/2$ consists of the sector abc with radius one and center angle 30° and the isosceles triangle Δacd with $|ac| = |ad| = 1$ and height corresponding to ac equal to $|bd|/2 = 1/2$. The case $R1/2$ was shown to be 4-universal but not 5-universal in [1]. Thus

$$A_4 \leq \text{area}(R1/2) = \text{area}(\text{sector } abc) + \text{area}(\Delta acd) = \pi/12 + 1/4 = 0.511\ldots.$$

Alt et al. [1] improved the upper bounds on A_4 with a better 4-universal case $S2$ depicted in Figure 13. $S2$ is constructed as follows. Let ab be a unit segment and take two circular arcs centered at a and b, respectively. Pick an arbitrary point c on the circular arc ob, and let $x = |bc|$. Draw a circle centered at c with radius x, let it intersect the arc oa at point d. Notice that d exists only if $|oc| \leq |cb|$. So c must be at the middle point of arc ob or higher, i.e., $\angle cab \geq 30°$. In Δabc, we have

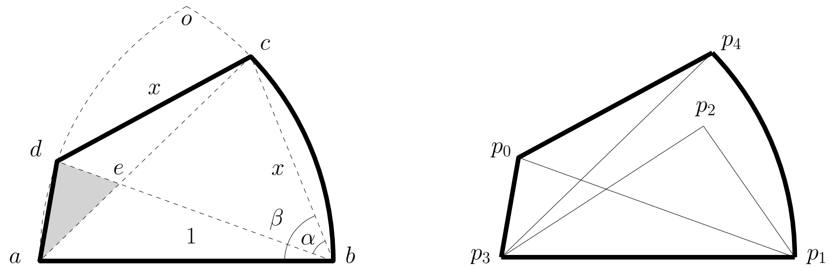

Figure 13: Left: convex 4-universal case $S2$ (in bold lines). Right: folding of a 4-link unit ruler $p_0p_1p_2p_3p_4$ into $S2$.

$|bc| = x$ and $|ac| = |ab| = 1$. Thus $|bc| = |cd| = x \geq \sqrt{2-\sqrt{3}} = 0.517\ldots$. Similar to the universal case C introduced in Section 2, $S2$ is proved to be 4-universal for any $x \in [\sqrt{2-\sqrt{3}}, 1]$. Notice that when $x = 1$, $S2$ becomes the 6-universal case $R1$, and when c is the midpoint of the arc ob, $S2$ is identical to $R1/2$.

The minimum area of $S2$ was claimed [1] to be less than 0.486. For completeness we provide full details in Section 4.1. Further, we observe that the shaded triangle Δade in Figure 13 (left) is not necessary for the folding algorithm introduced in [1]. If we discard this triangle, a family of nonconvex cases $abcde$ with parameter x is obtained. In Section 4.2, we prove that for $x = \sqrt{2-\sqrt{3}} = 0.517\ldots$,

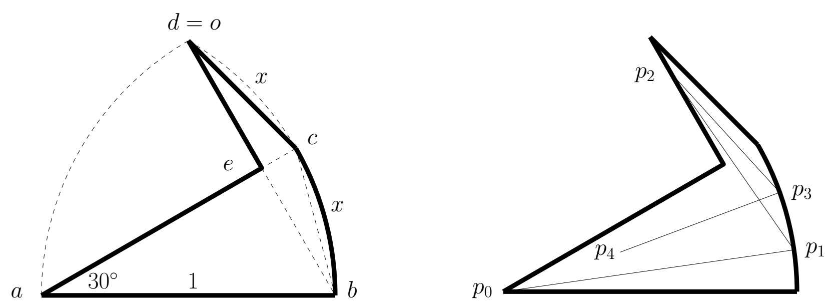

Figure 14: Left: nonconvex 4-universal case $C2$ (in bold lines). Right: folding of a 4-link unit ruler $p_0p_1p_2p_3p_4$ into $C2$.

i.e., c is the midpoint of arc ob, the nonconvex case $C2$ shown in Figure 14 has the smallest area in this family, $\pi/12 + \sqrt{7-4\sqrt{3}}/8 = 0.295\ldots$. We then show in Section 4.3 that $C2$ is a 4-universal but not 5-universal case. Thus, the upper bound on B_4 is improved from 0.486 to 0.296: $B_4 \leq 0.296$ (recall that $A_4 \leq 0.486$).

4.1 Area of $S2$

In preparation for calculating the area of $C2$, we provide the missing details for minimizing the area of $S2$. In Section 4.2, the area of $C2$ is derived based on the following calculations. The area of $S2$ is the sum of the areas of its three parts: the isosceles triangles Δbcd, Δabd and the circular segment cb. In Δbcd, $\alpha = \arccos\frac{1}{2x}$ and its area is $\frac{\sqrt{4x^2-1}}{4}$. In isosceles triangle Δabc, $\beta = \arccos\frac{x}{2}$ and its area is $\frac{x}{4}\sqrt{4-x^2}$. The area of the circular segment cb is the area of the circular sector abc minus the area of Δabc, i.e., $\frac{\pi}{2} - \arccos\frac{x}{2} - \frac{x}{4}\sqrt{4-x^2}$. In Δabd, $\angle abd = \beta - \alpha$, its area is

$\sin(\beta-\alpha)/2 = \sin\left(\arccos\frac{x}{2} - \arccos\frac{1}{2x}\right)/2$. In summary,

$$\begin{aligned}\text{area}(S2) &= \text{area}(\Delta bcd) + \text{area}(\Delta abd) + \text{area}(\text{circular segment } cb)\\ &= \frac{\sqrt{4x^2-1}}{4} + \sin\left(\arccos\frac{x}{2} - \arccos\frac{1}{2x}\right)/2 + \frac{\pi}{2} - \arccos\frac{x}{2} - \frac{x}{4}\sqrt{4-x^2}.\end{aligned}$$

Minimizing the area function on $x \in [0.5, 1]$ using Mathematica (see Appendix C) yields that for $x = 0.743$, we have $\text{area}(S2) \le 0.486$.

4.2 Area of $C2$

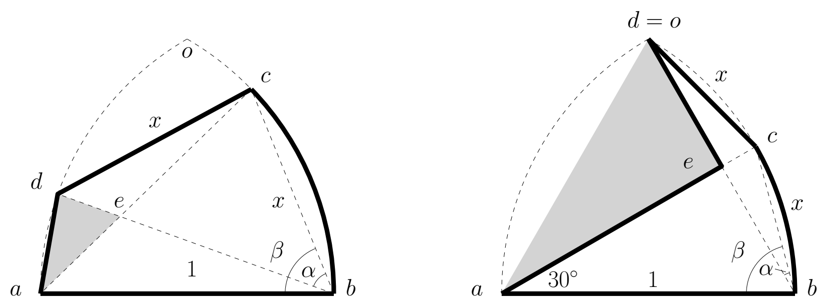

Figure 15: Obtaining $C2$ from $S2$. Left: $S2$ (in bold lines) with the shaded triangle discarded. Right: $C2$ (in bold lines) has minimum area at most 0.296 when c is the midpoint of the circular arc bd.

Due to the subtraction of triangle Δade, we need to calculate the area of triangle Δabe instead of Δabd which is used in the area formula of $S2$. In Δabc; see Figure 15 (left), $\angle bac = \pi - 2\beta$.

In Δabe, $\angle abe = \beta - \alpha$ and

$$\angle aeb = \pi - \angle bae - \angle abe = \pi - (\pi - 2\beta) - (\beta - \alpha) = \beta + \alpha.$$

So

$$\frac{|be|}{\sin(\pi - 2\beta)} = \frac{|ab|}{\sin(\beta+\alpha)} = \frac{1}{\sin(\beta+\alpha)}, \text{ or } |be| = \frac{\sin(2\beta)}{\sin(\beta+\alpha)}.$$

The area of Δabe is

$$\begin{aligned}\text{area}(\Delta abe) &= |ab| \cdot |be| \sin(\beta-\alpha)/2\\ &= \frac{\sin\left(2\arccos\frac{x}{2}\right)\sin\left(\arccos\frac{x}{2} - \arccos\frac{1}{2x}\right)}{\sin\left(\arccos\frac{x}{2} + \arccos\frac{1}{2x}\right)}.\end{aligned}$$

Notice that

$$\begin{aligned}\sin\left(2\arccos\frac{x}{2}\right) &= \frac{x\sqrt{4-x^2}}{2},\\ \sin\left(\arccos\frac{x}{2} - \arccos\frac{1}{2x}\right) &= \frac{\sqrt{4-x^2} - x\sqrt{4x^2-1}}{4x}\\ \sin\left(\arccos\frac{x}{2} + \arccos\frac{1}{2x}\right) &= \frac{\sqrt{4-x^2} + x\sqrt{4x^2-1}}{4x}.\end{aligned}$$

Thus, the area of the convex case $C2$ is

$$\begin{aligned}\text{area}(C2) &= \text{area}(\Delta bcd) + \text{area}(\text{circular segment } cb) + \text{area}(\Delta abe) \\ &= \frac{\sqrt{4x^2-1}}{4} + \frac{\pi}{2} - \arccos\frac{x}{2} - \frac{x}{4}\sqrt{4-x^2} + \frac{x(4-x^2) - x^2\sqrt{(4-x^2)(4x^2-1)}}{4\sqrt{4-x^2} + 4x\sqrt{4x^2-1}}.\end{aligned}$$

Solving for minimum value of the area function on $x \in [\sqrt{2-\sqrt{3}}, 1]$ using Mathematica (see Appendix D) gives that when $x = \sqrt{2-\sqrt{3}} = 0.517\ldots$, $C2$ has the minimum area. Since c is the midpoint of the circular arc bd, ce is perpendicular to ed and $|ed| = 1/2$, $|ce| = \sqrt{7-4\sqrt{3}}/2$. The minimum area of $C2$, depicted in Figure 15 (right), is

$$\text{area}(C2) = \text{area}(\text{sector } abc) + \text{area}(\Delta cde) = \pi/12 + \sqrt{7-4\sqrt{3}}/8 = 0.295\ldots.$$

4.3 $C2$ is 4-universal

First we show that any 3-link unit ruler $p_0p_1p_2p_3$ can be folded into $C2$.

Lemma 3. *$C2$ is a nonconvex 3-universal case.*

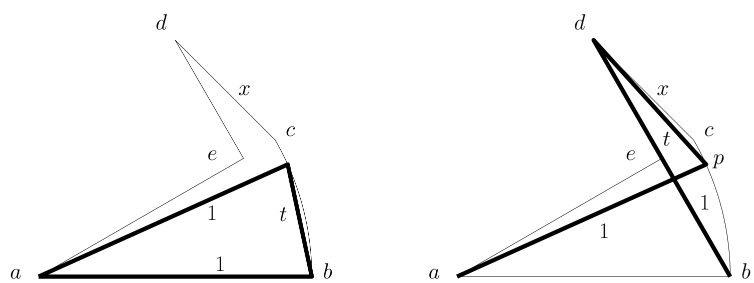

Figure 16: Folding of a 3-link ruler (in bold lines) into $C2$. Left: $t \le x$. Right: $t > x$.

Proof. It suffices to consider rulers with links $1, t, 1$ with $t \in (0, 1)$.

Case 1: t is small, i.e., $t \le x = \sqrt{2-\sqrt{3}}$. So both endpoints of the center link t can be placed on the circular arc bc; see Figure 16 (left). Since arc bc is centered at a with radius 1, both links of length one can be folded from arc bc to a.

Case 2: t is large, i.e., $|cd| = x < t \le 1 = |bd|$. As shown in Figure 16 (right), starting from point d, there exists a point p on arc bc such that $|dp| = t$. So the $1, t, 1$ ruler can be placed from b to d to p to a. □

Now we prove that any 4-link unit ruler $p_0p_1p_2p_3p_4$ can be folded inside $C2$.

Lemma 4. *$C2$ is a nonconvex 4-universal case.*

Proof. It suffices to consider unit rulers with links $1, t, t', 1$.

Case 1: $t + t' \le 1$. The folding problem for this ruler can be reduced to a folding problem for the 3-link ruler $1, t + t', 1$ which is already solved by Lemma 3.

Case 2: $t + t' > 1$. Without loss of generality, we can assume that $t \ge t'$. Label the endpoints of the ruler by p_i, $i = 0, 1, 2, 3, 4$, such that $|p_0p_1| = |p_3p_4| = 1$, $|p_1p_2| = t$ and $|p_2p_3| = t'$. Fold the ruler such that the first two links overlap each other, i.e., p_2 lies on p_0p_1. As illustrated in Figure 17 (left), we place p_0p_1 at bd, then p_2 is on eb otherwise $t+t' \leq 1$. $|p_2d| = t \geq t'$ and $|p_2b| = 1-t < t'$, so there exists a point p on the circular arc bcd (notice that the arc cd is imaginary) such that $|p_2p| = t'$.

Case 2a: p lies on arc bc. We can place p_3 at p then p_4 at a as illustrated in Figure 17 (left).

Case 2b: p lies on (the imaginary) arc cd. We flip the ruler around with respect to the axis ac. As shown in Figure 17 (right), now p_0 is at d, p_1 is at b and p_2 is at p_2'. Point p is also flipped to point p' on arc bc. So we can place p_3 at p' and p_4 at a.

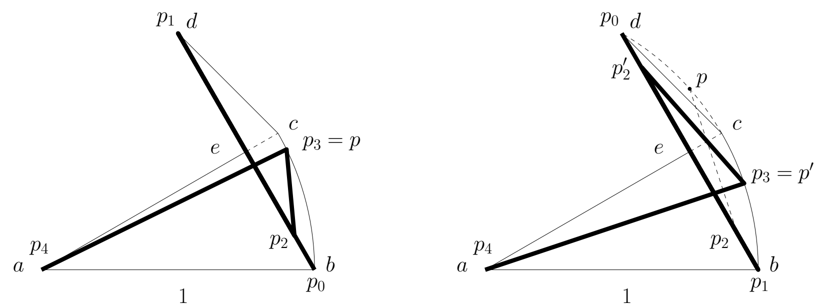

Figure 17: Folding of a 4-link ruler (in bold lines) into $C2$. Left: $|p_2c| \leq x$. Right: $|p_2c| > x$.

Thus, in both cases the 4-link ruler can be folded into $C2$. More formally, if p_0p_1 (or p_1p_0) is placed at bd and p_2 is folded on p_0p_1, we have $|p_2c| = \sqrt{(1/2-t)^2 + (7/4-\sqrt{3})}$. If $|p_2c| \geq t'$ (*Case 2a*), p_0 is placed at b. Otherwise $|p_2c| < t'$ (*Case 2b*), p_0 is placed at d. Then in both cases, there exists a point p on arc bc such that $|p_2p| = t'$. □

Hence Theorem 2 follows. Additionally we show the existence of a ruler with 5 links that cannot be folded inside $C2$.

Lemma 5. *$C2$ is not a 5-universal case.*

Proof. Consider folding the 5-link ruler 1, 0.6, 1, 0.6, 1 into $C2$. The 0.6 links are in between the 1 links so both their endpoints must lie on a, d or arc bc. But since $1 > 0.6 > \sqrt{2-\sqrt{3}}$, between d and some point on arc bc is the only possible position among all the combinations. Suppose the first 0.6 link is placed in this position, the other endpoints of the two 1 links adjacent to it must be placed at b and a respectively. But the second 0.6 link must also be placed in the same position which is impossible. □

5 Remarks

In Section 3, the best possible lower bound given by one 3-link ruler is achieved, whereas the one given by a 5-link ruler is not. Computer experiments suggest that 5-link rulers require folding area at least 0.137; more precisely:

- The minimum folding of a 5-link ruler with lengths 1, 0.6, 1, 0.6, 1 has (nonconvex) area at least 0.092.
- The minimum folding of a 5-link (symmetric) ruler with lengths $1, t, 1, t, 1$ has area at least 0.115 when $t = 0.8$.

- The minimum folding of a 5-link (asymmetric) ruler with lengths $1, t_1, 1, t_2, 1$ has area at least 0.137 when $t_1 = 0.7, t_2 = 0.4$.

The difficulty of approaching these better bounds lies in the complicated computations of nonconvex areas in many sub-cases. Note however that even the computational results were trusted and made rigorous, the resulting lower bounds would still be far away from the current upper bound of 0.583, which we believe is closer to the truth. Based on these observations, possible future research directions are deriving better lower bounds for universal cases using rulers with more links or using combinations of multiple rulers.

The problem of finding minimum area k-universal cases arose as another approach to the original problem of finding minimum area universal cases. Specifically, the following question is relevant:

1. Is there a k such that A_k (B_k) matches the universal convex (nonconvex) bound?

As mentioned earlier, Klein and Lenz [9] showed that no subset of $R2$ with a smaller area than $R2$ is a universal case. This is proved by using a ruler with n links where n goes to infinity. The authors showed that the only possible folding inside $R2$ covers the whole area of $R2$. Motivated by this result, we pose the following questions:

2. Does a similar result hold for the nonconvex universal case C?
3. Does a similar result hold for the nonconvex 4-universal case $C2$? Notice that in this problem, the method with rulers having the number of links going to infinity (which was used in [9]) is not applicable since the rulers have at most 4 links. However, a possible approach would be to use a combination of multiple rulers and show that no matter how the rulers are folded, $C2$ will be covered.

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A Minimum Area of C

```
In[1]:=
  c[x_]:=(1-2x+2x^2)ArcCos[(1-2x)/(2-2x)]+(2x-1)Sqrt[3-4x]/4
  FindMinimum[c[x], {x, 0, 0, 1}]

Out[2]= {0.582667, {x -> 0.165257}}
```

B Slope of p_0p_1

```
In[1]:= sol = NSolve[{(x-1)^2+y^2==1, (x-0.6Cos[16 Degree])^2+(y-0.6Sin[16
     Degree])^2==1, y<0}, {x, y}]

Out[2]= {{x -> 0.433945, y -> -0.824367}}

In[3]:= (y-0.6Sin[16 Degree])/(x-0.6Cos[16 Degree]) /. sol

Out[4]= {6.93043}
```

C Minimum Area of $S2$

```
In[1]:=
  s2[x_]:=Sqrt[4x^2-1]/4+Sin[ArcCos[x/2]-ArcCos[1/(2x)]]/2+
    Pi/2-ArcCos[x/2]-(x)Sqrt[4-x^2]/4
  FindMinimum[{s2[x], 1>=x>=0.5}, {x}]

Out[2]= {0.485502, {x -> 0.7439}}
```

D Minimum Area of $C2$

```
In[1]:=
  c2[x_]:=Sqrt[4x^2-1]/4+Pi/2-ArcCos[x/2]-(x)Sqrt[4-x^2]/4+
    (x(4-x^2)-x^2Sqrt[(4-x^2)(4x^2-1)])/(4Sqrt[4-x^2]+(4x)Sqrt[4x^2-1])
  FindMinimum[{c2[x], 1>=x>=Sqrt[2-Sqrt[3]]}, {x}]

Out[2]= {0.295293, {x -> 0.517638}}
```