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MAXIMUM AREA INDEPENDENT SETS IN DISK INTERSECTION GRAPHS

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Maximum Independent Set (MIS) and its relative *Maximum Weight Independent Set* (MWIS) are well-known problems in combinatorial optimization; they are NP-hard even in the geometric setting of unit disk graphs. In this paper, we study the *Maximum Area Independent Set* (MAIS) problem, a natural restricted version of MWIS in disk intersection graphs where the weight equals the disk area. We obtain: (i) Quantitative bounds on the maximum total area of an independent set relative to the union area; (ii) Practical constant-ratio approximation algorithms for finding an independent set with a large total area relative to the union area.

Keywords: Maximum independent set; disk intersection graph; approximation algorithm.

1. Introduction

Maximum Independent Set (MIS) is the problem of computing an independent set of maximum cardinality in a given undirected graph. In the weighted version, Maximum Weight Independent Set (MWIS), a non-negative weight is associated with each vertex of the graph, and the problem is to compute an independent set

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of maximum total weight. MWIS is often studied in geometric settings, where the input graph $G = (V, E)$ is the intersection graph of a set V of geometric objects. The *intersection graph* $G(\mathcal{S})$ of a set $\mathcal{S}$ of objects has a vertex representing each object in $\mathcal{S}$ and an edge between two vertices if and only if the corresponding objects intersect in their interiors [16]. For MWIS in geometric intersection graphs, a common choice for the weight is the volume (or area) of each geometric object. In this paper, we study the Maximum Area Independent Set (MAIS) problem, a restricted version of MWIS in disk intersection graphs where the weight equals the disk area. MAIS is a natural problem with many applications. We illustrate with one example: Given a set of potential wireless network servers at specified locations and with associated disk covering ranges, find a subset of servers that maximize the covered area under the constraint of non-interference. See also the paper by Alt *et al.* [2] for recent results on covering point sets by disks.

MIS in unit disk graphs is known to be NP-hard [9], which immediately implies the NP-hardness of MWIS and MAIS in general disk intersection graphs. A 3-approximation for MIS in unit disk graphs can be easily achieved by a sweep-line algorithm that repeatedly selects the leftmost disk disjoint from the previously selected disks [15]; for MIS in general disk intersection graphs, a 5-approximation can be obtained by repeatedly selecting the smallest disjoint disk instead [15], and can be extended to MWIS using the local-ratio technique [3]. In a seminal work by Hochbaum and Maass [14], a PTAS for MIS in unit disk graphs was obtained using a shifted grid strategy. Later, a PTAS for MWIS in general disk intersection graphs was found by Erlebach *et al.* [13] using a more sophisticated shifted hierarchical subdivision strategy. The time complexity of the PTAS by Erlebach *et al.* [13] was $n^{O(1/\varepsilon^2)}$, which was improved to $n^{O(1/\varepsilon)}$ by Chan [8]. Both approximation schemes are relatively complicated, with running times essentially impractical for larger n; so their main merit is largely theoretical, rather than practical. Chan [8] remarked that "much work remains in order to develop truly practical approximation algorithms."

We first introduce some notations. Denote by $|C|$ the area (i.e., the Lebesgue measure) of a convex set C in $\mathbb{R}^2$. For a set $\mathcal{S}$ of convex sets in $\mathbb{R}^2$, define its *union area* as $|\mathcal{S}| = |\cup_{C \in \mathcal{S}} C|$. For two points a and b, denote by $|ab|$ the length of the segment ab.

Two natural questions can be asked about the MAIS problem in disk intersection graphs:

(1) How large is the maximum total area of an independent set relative to the union area?
(2) How to compute efficiently an independent set with a large total area relative to the union area?

Question 1 above is in fact related to "the Rados' problem on selecting disjoint squares" [10] , first posed by T. Rado [24] in 1928, subsequently studied by many researchers, and considered in detail by R. Rado [21,22,23] in a more general setting for various classes of convex sets. Let $\mathcal{S}$ be a set of homothetic copies of a convex set S, assumed to be compact and with nonempty interior. For the case that $\mathcal{S}$ is a set of congruent disks, R. Rado [21] showed[a] that there exists an independent set $\mathcal{I} \subseteq \mathcal{S}$ such that $|\mathcal{I}|/|\mathcal{S}| \geq \frac{\pi}{8\sqrt{3}} > \frac{1}{4.4107}$. For the case that $\mathcal{S}$ is a set of axis-aligned squares, T. Rado [24] observed that a greedy algorithm, which repeatedly selects the largest square disjoint from those previously selected, can find an independent set $\mathcal{I} \subseteq \mathcal{S}$ such that $|\mathcal{I}|/|\mathcal{S}| \geq 1/9$. When the squares in $\mathcal{S}$ are congruent, it is known [10] that there exists an independent set $\mathcal{I} \subseteq \mathcal{S}$ such that $|\mathcal{I}|/|\mathcal{S}| \geq 1/4$. This bound 1/4 is clearly the best possible: take four unit squares sharing a common vertex, only one of them can be in an independent set. T. Rado [24] conjectured, in 1928, that the bound 1/4 also holds for arbitrary, i.e., not necessarily congruent, axis-aligned squares. Forty-five years later, in 1973, an ingenious construction by Ajtai [1] with several hundred squares disproved T. Rado's conjecture! In the spirit of T. Rado's conjecture for squares [24], we make the following conjecture for disks:

Conjecture 1. *For any set $\mathcal{S}$ of closed disks in the plane, there is a subset $\mathcal{I} \subseteq \mathcal{S}$ of pairwise-disjoint disks such that $|\mathcal{I}|/|\mathcal{S}| \geq 1/4$.*

The subset $\mathcal{I}$ of pairwise-disjoint disks is an independent set in the intersection graph $G(\mathcal{S})$. Alternatively, $\mathcal{I}$ can be interpreted as a packing of disks with positions restricted to $\mathcal{S}$. Geometric packing and covering are notoriously hard problems; a lot of research has been done in the past to find the densest packing, the thinnest covering, and the packing-covering ratio of disks in the plane and balls in the space [17,7]. Our conjecture is about disks in the plane, but a general question can be asked for balls in any dimension d: What is the largest value ξ_d such that, for any set $\mathcal{S}$ of balls in $\mathbb{R}^d$, there is always an independent set $\mathcal{I} \subseteq \mathcal{S}$ with $|\mathcal{I}|/|\mathcal{S}| \geq \xi_d$? To be precise, define

$$\xi_d = \inf_{\mathcal{S}} \sup_{\mathcal{I}} \frac{|\mathcal{I}|}{|\mathcal{S}|},$$

where $\mathcal{S}$ ranges over all sets of balls in $\mathbb{R}^d$, and $\mathcal{I}$ ranges over all independent subsets of $\mathcal{S}$. Then our conjecture is that $\xi_2 \geq 1/4$. Rado [21] observed an upper bound $\xi_d \leq 1/2^d$: Let $\mathcal{S}$ be the set of all unit balls in $\mathbb{R}^d$ that contain a common point (say, the origin). The intersection graph $G(\mathcal{S})$ is a clique, so any independent set $\mathcal{I}$ can include at most one unit ball. The bound follows by comparing the two volumes $|\mathcal{I}|$ and $|\mathcal{S}|$. Moreover, one can get arbitrarily close to the ratio $1/2^d$ with finite sets of balls.

For the easy case $d = 1$, a lower bound $\xi_1 \geq 1/2$ is known: for any set of closed intervals $\mathcal{S}$ on the real line, there is a subset $\mathcal{I} \subseteq \mathcal{S}$ of disjoint intervals such that the total length of the intervals in $\mathcal{I}$ is at least 1/2 of the total length of the union of the intervals in $\mathcal{S}$. This fact was first noted by R. Rado [21]. Nevertheless, for

[a] According to R. Rado, this result was first found by Besicovitch but not published, and later rediscovered by himself.

comparison with our proofs for the case $d = 2$, we include in the following a short proof for $\xi_1 \geq 1/2$ rediscovered by ourselves:

Without loss of generality, assume that no interval in $\mathcal{S}$ is completely covered by the union of other intervals: otherwise remove any such interval and the union length $|\mathcal{S}|$ remains the same. Observe that any point on the line can intersect at most two intervals in $\mathcal{S}$: suppose that a point intersects three intervals, then the union of two of them, the one with the left-most endpoint and the one with the right-most endpoint, would cover the third, a contradiction. Order the intervals by their left endpoints, then color the ith interval by $i \pmod 2$. We obtain a two-coloring of the intersection graph $G(\mathcal{S})$. The intervals in $\mathcal{S}$ can be partitioned into two independent sets, one of which has a total length of at least $|\mathcal{S}|/2$.

The lower bound $\xi_1 \geq 1/2$ and the upper bound $\xi_d \leq 1/2^d$ together imply an exact bound $\xi_1 = 1/2$ for the case $d = 1$. This is indeed the only known case for which we have matching lower and upper bounds.

Our Results. In this paper, we derive new bounds on the value ξ_2 and its variants. The underlying set $\mathcal{S}$ of disks in the definition of ξ_2 may be restricted in various ways: $\rho_{\text{arbitrary}} = \xi_2$ for arbitrary disks, ρ_{clique} for pairwise-intersecting disks (cliques), ρ_{disjoint} for interior-disjoint (non-overlapping) disks, and ρ_{unit} for unit disks. We also provide efficient algorithms that, given a finite set $\mathcal{S}$ of closed disks in the plane, compute an independent subset $\mathcal{I}$ with a large total area $|\mathcal{I}|$ relative to the union area $|\mathcal{S}|$.

R. Rado [21] showed that $\rho_{\text{unit}} \geq \frac{\pi}{8\sqrt{3}} > 1/4.4107$. This remains the current best lower bound for ρ_{unit}. Here we give a very simple $O(n \log n)$ time algorithm achieving a looser bound $\rho_{\text{unit}} \geq 1/(5 + 4/\pi) > 1/6.2733$, and a linear-time approximation scheme achieving (approximately) R. Rado's bound:

Theorem 1. *Let $\mathcal{S}$ be a set of n closed unit disks in the plane. Then,*

(i) *An independent set of area $|\mathcal{S}|/\lambda$, where $\lambda = 5+4/\pi < 6.2733$, can be computed in $O(n \log n)$ time;*
(ii) *For any given $\varepsilon > 0$, an independent set of area $|\mathcal{S}|/(\lambda+\varepsilon)$, where $\lambda = 8\sqrt{3}/\pi < 4.4107$, can be computed in $O(n/\varepsilon^2)$ time.*

The previous best bound for arbitrary disks is $\rho_{\text{arbitrary}} \geq (1 + 1/200704)/9 \approx 1/8.999955$, which is implicit from a more general result of R. Rado [21] for centrally-symmetric convex sets. Here we derive an improved bound:

Theorem 2. *Let $\mathcal{S}$ be a set of n closed disks in the plane. Then an independent set of area $|\mathcal{S}|/\lambda$, where $8.4897 < \lambda < 8.4898$, can be computed in $O(n^2)$ time. That is, $\rho_{\text{arbitrary}} > 1/8.4898$.*

For comparison, the current best bound for axis-aligned squares in the plane is $1/8.6$, due to Zalgaller [27]. We revisit this and many other variants of Rados' problem in a forthcoming paper [4].

It is not hard to show that Conjecture 1 holds for the special case of pairwise-intersecting disks. The diameter of a set $\mathcal{S}$ of pairwise-intersecting closed disks in the plane is at most two times the maximum disk diameter. It is well-known [25] that the area of a planar set is at most the area of a disk of the same diameter. Therefore, the area of the largest disk in $\mathcal{S}$ is at least $1/4$ of the union area $|\mathcal{S}|$. Let the independent set $\mathcal{I}$ include only the largest disk in $\mathcal{S}$, and we have $|\mathcal{I}|/|\mathcal{S}| \geq 1/4$. This lower bound is tight, compared to the upper bound $\xi_2 \leq 1/4$ mentioned earlier. We thus have:

Proposition 1. *Let $\mathcal{S}$ be a set of pairwise-intersecting closed disks $\mathcal{S}$ in the plane. Then there is a disk in $\mathcal{S}$ whose area is at least $|\mathcal{S}|/4$. That is, $\rho_{\text{clique}} = 1/4$.*

Consider now the variant of intersection graph in which a pair of objects sharing only boundary points are also considered intersecting. For the case of interior-disjoint disks, i.e., the disks may be tangent but do not overlap in the interior, the intersection (tangency) graph is planar and hence 4-colorable, as pointed out by Pollack [19]. We therefore have a $1/4$ lower bound on the area ratio. We mention here a related question [18] asked by Erdős in 1983: What is the largest number $F = F(n)$ with the property that every set of n non-overlapping unit disks in the plane has an independent subset with at least F members? Since for non-overlapping disks (unit or not) the intersection graph is 4-colorable, where each color class forms an independent set, the largest color class has at least $n/4$ members; therefore $F(n) \geq \lceil n/4 \rceil$. Csizmadia [11] improved this lower bound to $\lceil 9n/35 \rceil$. The best current upper and lower bounds [7] are due to Pach and Tóth [18], and Swanepoel [26], respectively:

$$\left\lceil \frac{8n}{31} \right\rceil \leq F(n) \leq \left\lceil \frac{5n}{16} \right\rceil \approx 0.3125n.$$

The upper bound, which holds for sufficiently large n, immediately gives an upper bound on the area ratio for non-overlapping disks: $\rho_{\text{disjoint}} \leq 5/16 = 0.3125$. One can use this construction or a previous one due to Chung, Graham and Pach [12], then add a number of smaller equal disks to get a slightly better upper bound; we discuss this briefly in Section 4. However our best construction is of a different type and uses an infinite number of disks of different radii; again, a finite small subset of disks in this construction gets arbitrarily close to this bound, i.e., below 0.3028.

Theorem 3. *Let $\mathcal{S}$ be a set of non-overlapping closed disks in the plane. Then there is a subset $\mathcal{I} \subseteq \mathcal{S}$ of disjoint disks whose total area is at least $|\mathcal{S}|/4$, and there are examples where no such subset has area more than $0.3028|\mathcal{S}|$. That is, $1/4 \leq \rho_{\text{disjoint}} < 0.3028$.*

2. Unit disks: proof of Theorem 1

In this section we discuss two different approaches for finding an independent set $\mathcal{I}$ of large total area in a given set $\mathcal{S} = \{D_1, D_2, \ldots, D_n\}$ of unit disks. Denote by c_i the center of a disk D_i.

Algorithm U1 The first approach applies the well-known sweep-line technique. It is interesting to note that, long before the sweep-line technique was introduced in computational geometry, it had already been used implicitly by R. Rado [21] in obtaining lower bounds on the volume ratios of independent sets in systems of congruent homothetic convex bodies. We review this technique here and show that a tighter analysis is possible for unit disks. We also choose to discuss the technique because it is very simple and can be implemented efficiently. To construct an independent set $\mathcal{I} \subseteq \mathcal{S}$, initialize $\mathcal{I}$ to be empty, then repeat the following selection step until $\mathcal{S}$ is empty:

> Let D_i be the leftmost disk in $\mathcal{S}$. Let $\mathcal{S}_i \subseteq \mathcal{S}$ be the set of disks that intersect D_i ($D_i \in \mathcal{S}_i$). Add D_i to $\mathcal{I}$, then remove from $\mathcal{S}$ the disks in $\mathcal{S}_i$.

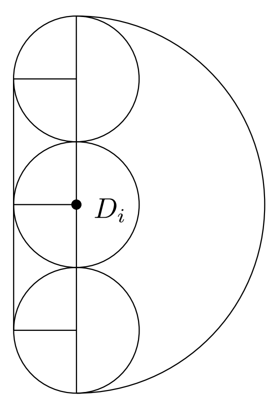

Fig. 1. Bound for the union area $|\mathcal{S}_i|$ in algorithm U1.

From Fig. 1, it is easy to see that

$$|\mathcal{S}_i| \leq 2 \cdot \frac{\pi}{4} + 4 + \frac{\pi}{2} \cdot 3^2 = 5\pi + 4.$$

It follows that

$$\frac{|D_i|}{|\mathcal{S}_i|} \geq \frac{\pi}{5\pi + 4}.$$

We therefore have the bound $|\mathcal{I}|/|\mathcal{S}| \geq 1/\lambda$, where $\lambda = 5 + 4/\pi < 6.2733$.

Algorithm U1 can be implemented in $O(n \log n)$ time as follows. We sort the centers of the disks by x-coordinate. Let $x = x_0$ be the sweeping line. We maintain the set $\mathcal{I}'$ of disks in $\mathcal{I}$ whose centers have x-coordinates at least $x_0 - 1$. The disks in $\mathcal{I}'$ are stored in a balanced binary search tree with y-coordinates of their centers as keys. When the sweeping line hits the center of a disk D_i we locate the y-coordinate of $c_i = (x, y)$ in $\mathcal{I}'$ in $O(\log n)$ time. Suppose for simplicity that the disks in $\mathcal{I}'$ are $D_1, D_2, \ldots, D_k$. Suppose that y is between y-coordinates of c_j and c_{j+1}.

We check whether D_i intersects at least one of eight disks $D_{j-3}, D_{j-2}, \ldots, D_{j+4}$. If D_i intersects a disk D_k, then D_i belongs to $\mathcal{S}_k$ and we do not add it to $\mathcal{I}$. Otherwise none of the disks of $\mathcal{I}$ intersects D_i, and we add it to $\mathcal{I}$ and $\mathcal{I}'$. Clearly, the total running time is $O(n \log n)$.

Algorithm U2 The second approach is based on a lattice technique and goes back to another old result of R. Rado [21] , who proved the existence of a subset $\mathcal{I}$ of disjoint disks in $\mathcal{S}$ such that $|\mathcal{I}|/|\mathcal{S}| \geq 1/\lambda$, where $\lambda = 8\sqrt{3}/\pi < 4.4107$.

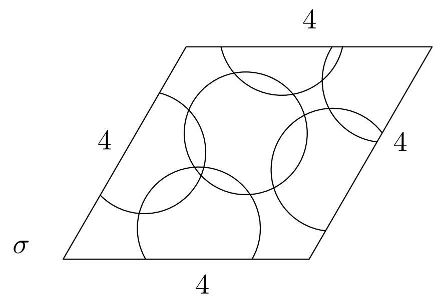

Fig. 2. The rhombus σ.

R. Rado's proof is as follows (it is worth comparing the argument with that for the case $d = 1$). Consider the equilateral triangular lattice Λ determined by the two bases $u = (4, 0)$ and $v = (2, 2\sqrt{3})$: two adjacent lattice points are at distance 4. Fix an arbitrary cell σ (a rhombus of side length 4) of the lattice. See Fig. 2. The area of σ is $|\sigma| = 8\sqrt{3}$. Translate each cell of the lattice to overlap with σ. There is a point in $p \in \sigma$ that is covered at least $k = \lceil |\mathcal{S}|/|\sigma| \rceil$ times. The original points covered by the union of the disks which correspond to p have pairwise distances at least 4. Pick an arbitrary disk in $\mathcal{S}$ covering each such point. Observe that these disks are pairwise disjoint in their interior, therefore $\mathcal{S}$ contains an independent set of size k with total area $\pi k = \pi \lceil |\mathcal{S}|/|\sigma| \rceil \geq |\mathcal{S}|/\lambda$, where $\lambda = |\sigma|/\pi = 8\sqrt{3}/\pi < 4.4107$.

We first note that R. Rado's proof can be turned into $O(n^2)$ time algorithm if we compute the arrangement of circular arcs intersecting σ. In particular, the algorithm finds a point in σ which is covered the maximum number of times (in all translated cells in the lattice) by the union of the disks. Recently, Braß *et al.* [6] studied a colored version of the maximum area independent set problem, and noticed its connection with the following 3SUM-hard problem: Given a set of n unit radius discs in the plane, decide whether there is a point of depth k, i.e., a point covered by k disks. This may indicate the optimality of the above $O(n^2)$ time algorithm that implements R. Rado's proof.

We now show that a $(1+\varepsilon)$-approximation of R. Rado's bound can be achieved by our second algorithm, U2, in linear time. Note that algorithm U2 makes use of the floor function; in contrast, algorithm U1 strictly follows the standard RAM machine model.

Let $\varepsilon > 0$ be a constant, and let Λ be the triangular lattice used in R. Rado's proof. Consider the lattice Λ/k (with cell size scaled by $1/k$) where k is a positive integer that will specified later. For every disk $D_i \in \mathcal{S}$, we compute the set of (small) triangles of the lattice Λ/k that are contained in D_i, and let $\mathcal{T}$ be set of all such triangles. Clearly, $\mathcal{T}$ has $O(nk^2)$ triangles, since each disk D_i can only intersect $O(1)$ triangles in the lattice Λ.

The triangles contained in a disk D_i cover a concentric disk D'_i of radius $1-1/k$. The set of disks $\{D'_1, \ldots, D'_n\}$ can be obtained by first scaling the set $\{D_1, \ldots, D_n\}$ with a factor of $1-1/k$ then uniformly increasing the pairwise distances of the disk centers with a factor of $1/(1-1/k)$. The scaling changes the union area to $(1-1/k)^2|\mathcal{S}|$. Increasing pairwise distances does not decrease the union area [5]. Therefore the total area of the triangles in $\mathcal{T}$ (contained in some D_i) is at least $(1-1/k)^2|\mathcal{S}|$. Translate now all triangles in $\mathcal{T}$ to σ as in R. Rado's proof. Let $\mathcal{T}_\sigma$ be the set of $2k^2$ small triangles in σ. For each triangle in $\mathcal{T}_\sigma$ we compute the number of times it is covered by triangles in $\mathcal{T}$ (from different cells of Λ). This takes $O(nk^2)$ time. There is a triangle in $\mathcal{T}_\sigma$ covered at least $j = \lceil (1-1/k)^2|\mathcal{S}|/|\sigma| \rceil$ times, and let p be one of its vertices. The original points covered by the union of the disks that correspond to p have pairwise distances at least 4. Pick an arbitrary disk in $\mathcal{S}$ covering each such point. Note that these disks are interior-disjoint. Therefore $\mathcal{S}$ contains an independent set of m disks with total area

$$\pi m = \pi \left\lceil \left(1 - \frac{1}{k}\right)^2 \frac{|\mathcal{S}|}{|\sigma|} \right\rceil \geq \left(1 - \frac{1}{k}\right)^2 \frac{|\mathcal{S}|}{\lambda} \geq \frac{|\mathcal{S}|}{\lambda + \varepsilon},$$

if $k = O(1/\varepsilon)$. The total running time is $O(nk^2) = O(n/\varepsilon^2)$.

3. Arbitrary disks: proof of Theorem 2

Let $\mathcal{S} = \{D_1, D_2, \ldots, D_n\}$ be a set of disks. For a disk D_i in $\mathcal{S}$, denote by $\mathcal{S}_i$ the set of disks in $\mathcal{S}$ that intersect D_i ($D_i \in \mathcal{S}_i$). Denote by r_i and c_i, respectively, the radius and the center of D_i.

Let D_l be the largest disk in $\mathcal{S}$. Let D'_l be the disk of radius 3 concentric with D_l. Observe that all disks in $\mathcal{S}_l$ are contained in D'_l. Therefore, an easy bound of $|\mathcal{I}|/|\mathcal{S}| \geq 1/9$ can be achieved by a greedy algorithm [21], which finds an independent set $\mathcal{I}$ by repeatedly adding to $\mathcal{I}$ the largest disk in $\mathcal{S}$ disjoint from all previously added disks. In general, this greedy algorithm achieves a volume ratio of $1/3^d$ for any dimension $d \geq 1$. For comparison, we note that, to the best of our knowledge, the current best volume ratio [21] for balls in dimension d is

$$|\mathcal{I}|/|\mathcal{S}| \geq (1 + 7^{-d}(d+2)^{-d^2-d})/3^d,$$

which is very close to the trivial bound of $|\mathcal{I}|/|\mathcal{S}| \geq 1/3^d$.

We now describe an algorithm that achieves an area ratio of $1/\lambda$, where $8.4897 < \lambda < 8.4898$ (λ will be defined later). To construct an independent set $\mathcal{I} \subset \mathcal{S}$, initialize $\mathcal{I}$ to be empty, then repeat the following selection step until $\mathcal{S}$ is empty:

Find the largest disk D_l in $\mathcal{S}$. Without loss of generality, assume[b] that $r_l = 1$. Find two disks D_i and D_j in $\mathcal{S}_l$ such that the diameter δ of their union is maximum. If $\delta \le 2\sqrt{\lambda}$, add D_l to $\mathcal{I}$, and then remove from $\mathcal{S}$ the disks in $\mathcal{S}_l$. Otherwise, add D_i and D_j to $\mathcal{I}$ (we will show that D_i and D_j are necessarily disjoint in this case), and then remove from $\mathcal{S}$ the disks in $\mathcal{S}_i \cup \mathcal{S}_j$.

The parameter λ will be chosen to balance two cases. We use again the fact that the maximum area of a planar set of diameter δ is at most $\pi\delta^2/4$, the area of a disk of the same diameter [25]. So if $\delta \le 2\sqrt{\lambda}$ (the first case), then we have

$$|D_l| = \pi \ge (\pi\delta^2/4)/\lambda \ge |\mathcal{S}_l|/\lambda.$$

Now suppose that $\delta > 2\sqrt{\lambda}$ in the second case. We will show that the two disks D_i and D_j are disjoint and satisfy the following inequality:

$$|D_i| + |D_j| \ge |\mathcal{S}_i \cup \mathcal{S}_j|/\lambda.$$

If the two disks intersect, then we would have $\delta \le 2r_i + 2r_j \le 4$, contradicting our assumption that $\delta > 2\sqrt{\lambda} > 4$. Therefore D_i and D_j must be disjoint. We now bound $r_i + r_j$ in terms of λ:

$$2\sqrt{\lambda} < \delta = r_i + |c_ic_j| + r_j \le r_i + |c_ic_l| + |c_jc_l| + r_j \le r_i + (r_i + 1) + (r_j + 1) + r_j,$$

$$r_i + r_j > \sqrt{\lambda} - 1. \tag{1}$$

Put $r_{\min} = \sqrt{\lambda} - 2$. Both r_i and r_j are at most 1 and at least $r_{\min}$.

Let D'_i be the disk of radius $r_i + 2$ concentric with D_i. Let D'_j be the disk of radius $r_j + 2$ concentric with D_j. The intersection $D'_i \cap D'_j$ contains D_l, and consists of two caps: a cap of D'_i and a cap of D'_j. The total height of the two caps is at least the diameter of D_l, which is 2. Denote by $A(R, h)$ the area of a cap of radius R and height h. It is well known [28] that

$$A(R, h) = R^2 \arccos(1 - h/R) - (R - h)\sqrt{R^2 - (R - h)^2}.$$

We clearly have

$$|D'_i \cap D'_j| \ge 2A(r_{\min} + 2, 1) = 2A(\sqrt{\lambda}, 1).$$

The disks in $\mathcal{S}_i \cup \mathcal{S}_j$ are contained in $D'_i \cup D'_j$, so we have

$$|\mathcal{S}_i \cup \mathcal{S}_j| \le |D'_i \cup D'_j| = |D'_i| + |D'_j| - |D'_i \cap D'_j|.$$

[b]This assumption mainly simplifies the analysis, and is not implemented in the algorithm.

Therefore,

$$\frac{|\mathcal{S}_i \cup \mathcal{S}_j|}{|D_i| + |D_j|} \le \frac{\pi(r_i+2)^2 + \pi(r_j+2)^2 - 2A(\sqrt{\lambda},1)}{\pi(r_i^2+r_j^2)}$$

$$= 1 + \frac{4(r_i+r_j) + 8 - 2A(\sqrt{\lambda},1)/\pi}{r_i^2+r_j^2}$$

$$\le 1 + \frac{4(r_i+r_j) + 8 - 2A(\sqrt{\lambda},1)/\pi}{(r_i+r_j)^2/2}.$$

Put

$$f(\lambda) = 8 - 2A(\sqrt{\lambda},1)/\pi.$$

If we choose λ such that $f(\lambda) \ge 0$, then the function

$$g(x) = 1 + \frac{4x + f(\lambda)}{x^2/2}$$

would be decreasing for $x > 0$, and from (1) we would have

$$\frac{|\mathcal{S}_i \cup \mathcal{S}_j|}{|D_i| + |D_j|} \le g(\sqrt{\lambda} - 1).$$

Define λ to be the solution of the equation

$$g(\sqrt{\lambda} - 1) = \lambda.$$

A calculation shows that $8.4897 < \lambda < 8.4898$, and that $f(\lambda) \approx 6.0599 \ge 0$. Therefore

$$\frac{|\mathcal{S}_i \cup \mathcal{S}_j|}{|D_i| + |D_j|} \le g(\sqrt{\lambda} - 1) = \lambda.$$

Thus, the above algorithm computes an independent set $\mathcal{I}$ of $\mathcal{S}$ such that $|\mathcal{I}| \ge |\mathcal{S}|/\lambda$.

We now show how to implement the algorithm in $O(n^2)$ time. We perform some preprocessing before the selection steps. For each disk D_k, construct a circular list $\mathcal{S}_k$ of the other disks that intersect it; the disks in $\mathcal{S}_k$ are ordered by the directions of the vectors from the center of D_k to their centers. This can be done in $O(n^2)$ time by computing the arrangement of the lines $\{\ell_k \mid D_k \in \mathcal{S}\}$ dual to the disk centers $\{c_k \mid D_k \in \mathcal{S}\}$, since the circular order of the other disk centers around a disk center c_k corresponds to the linear order of intersections of the other dual lines with the dual line ℓ_k.

We next consider each selection step. The largest disk D_l can be found in $O(n)$ time. To find the two disks D_i and D_j, first construct the convex hull of the disks in $\mathcal{S}_l$ using a variant of Graham scan, then apply the standard rotating calipers algorithm [20]. This can be done in $O(n)$ time since the list $\mathcal{S}_l$ is in circular order. To remove a disk from the circular lists, simply mark the disk "removed" and defer the actual removal until the convex hull construction of a later step. Since each disk is removed at most once from each list, the total time of all such removals is $O(n^2)$.

There are at most n selection steps. So the total running time of the algorithm is $O(n^2)$.

4. Non-overlapping disks: proof of Theorem 3

We first review two constructions with unit disks and describe how they can be adapted for our purpose, i.e., for obtaining better estimates for arbitrary disks. Consider first the construction of Chung, Graham and Pach [12,17], which consists of 19 unit (large) disks arranged as in Fig. 3. Add to it 12 smaller equal disks, each tangent to three unit disks, as shown in the figure. By repeating this group of 29 disks, one can get arbitrarily large sets of disks attaining the same bound.

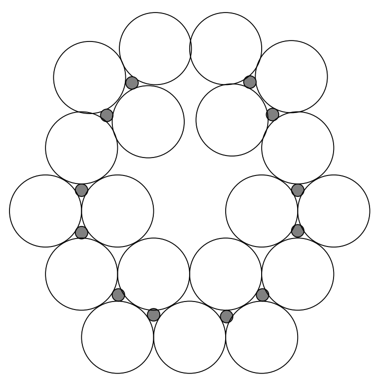

Fig. 3. Nineteen unit disks and twelve smaller disks of radius $2/\sqrt{3} - 1 \approx 0.1547$.

The small disks have radius $x = 2/\sqrt{3}-1$. It is known [12,17] that any independent set cannot contain more that six large disks. To these six large disks one can add one small disk. One can now check that an independent set containing two or more small disks cannot contain more than five large disks, therefore

$$\rho_{\text{disjoint}} \le \frac{6 + x^2}{19 + 12x^2} \approx 0.3123 .$$

Note that this bound is only very slightly smaller than the best current bound for unit disks [18], namely $5/16 = 0.3125$. In a similar way, by adding a number of small congruent disks each tangent to one of the groups of three unit disks forming equilateral triangles in the intersection graph for the $5/16$ construction of Pach and Tóth [18], one can get an even better bound:

$$\rho_{\text{disjoint}} \le \frac{10 + 2x^2}{32 + 22x^2} \approx 0.3089 .$$

We now give the details for our best construction depicted in Fig. 4. It uses two infinite decreasing sequences of radii converging to 0: $y_1 > y_2 > \ldots$ and $z_1 > z_2 > \ldots$; the choice of these values will be explained below. Consider a unit radius disk

D_0 surrounded by nine disks of radius x which are tangent to D_0, and so that every two consecutive disks are tangent. We have

$$x = \frac{\sin(\pi/9)}{1 - \sin(\pi/9)} \approx 0.5198 \, .$$

We add one inner layer of nine smaller disks of radius y_1, where each such disk is tangent to D_0 and to two consecutive disks of radius x. We extend the inner layer with nine smaller disks of radius y_2, where each such disk is tangent to a disk of radius y_1 and to two consecutive disks of radius x. In this way, we add an infinite number of disks (which get smaller and smaller) in between any two consecutive disks of radius x, and converging to the tangency points of the disks of radius x. Note that the values of the y-sequence are determined (by x and 1).

Let $z_1 < x$ be a value to be specified later. We add one outer layer of nine smaller disks of radius z_1, where each such disk is tangent to two consecutive disks of radius x. We extend the outer layer with nine smaller disks of radius z_2, where each such disk is tangent to a disk of radius z_1 and to two consecutive disks of radius x. In this way, we add an infinite number of disks (which get smaller and smaller) in between any two consecutive disks of radius x, and converging to the tangency points of the disks of radius x.

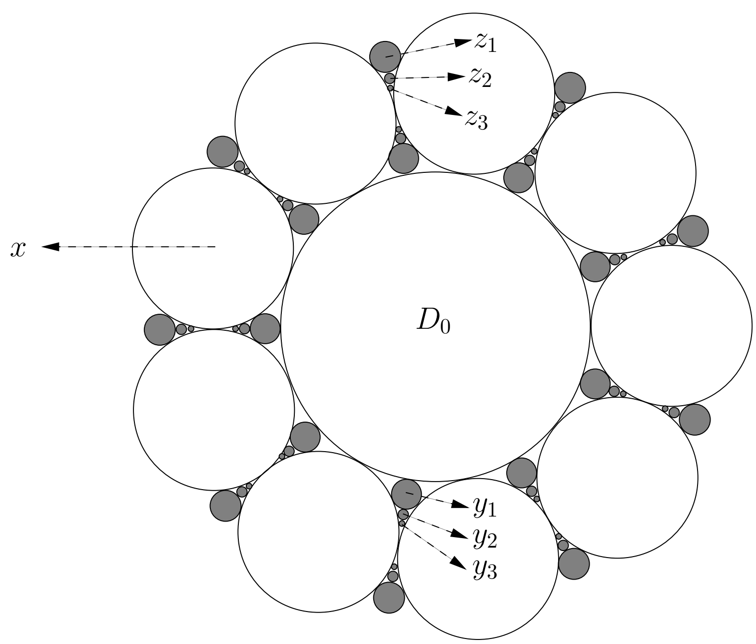

Fig. 4. A unit disk, nine disks of radius $x \approx 0.5198$, nine infinite sequences of smaller disks of radii $y_1 > y_2 > \ldots$, where $y_1 \approx 0.0967$, $y_2 \approx 0.0365$, etc., and nine infinite sequences of smaller disks of radii $z_1 > z_2 > \ldots$, where $z_1 \approx 0.0956$, $z_2 \approx 0.0363$, etc.

By Descartes circle theorem, the radius y_1 of the first (largest) circle in the inner layer is

$$y_1 = \frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3 + 2\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}}$$

for $r_1 = 1$, and $r_2 = r_3 = x$, and this yields $y_1 \approx 0.0967$. Note that using the above formula, y_{i+1} can be easily computed from y_i for each i. Similarly, once z_1 is selected, z_{i+1} can be easily computed from z_i for each i using the same formula.

Consider an independent set $\mathcal{I}$ of disks. If $D_0 \in \mathcal{I}$, no disk of radius x or radius y_1 can belong to $\mathcal{I}$. Since the radii of disks in each sequence are rapidly decreasing, ρ_{disjoint} cannot exceed

$$q_1 = \frac{1 + 9(z_1^2 + z_3^2 + \ldots) + 9(y_2^2 + y_4^2 + \ldots)}{1 + 9(x^2 + \sum_{i=1}^{\infty} y_i^2 + \sum_{i=1}^{\infty} z_i^2)}.$$

If $D_0 \notin \mathcal{I}$, at most four disks of radius x can belong to $\mathcal{I}$, together with at most one subsequence of disks with radii $y_1, y_3, \ldots$ and one subsequence of disks with radii $z_1, z_3, \ldots$ (this is the combination with the maximum total area), thus ρ_{disjoint} cannot exceed

$$q_2 = \frac{4x^2 + (y_1^2 + y_3^2 + \ldots) + (z_1^2 + z_3^2 + \ldots)}{1 + 9(x^2 + \sum_{i=1}^{\infty} y_i^2 + \sum_{i=1}^{\infty} z_i^2)}.$$

We can approximately balance the two ratios ($q_1 \approx q_2$) using a computer calculation. For instance, by keeping only 10 disks in each sequence, and setting $z_1 = 0.09567$, we obtain $q_1, q_2 < 0.3028$, and consequently $\rho_{\text{disjoint}} < 0.3028$. By keeping more disks — say 50, or even the whole infinite sequence — one can improve the digit accuracy of the ratio, but the bound still remains above 0.3027. This concludes the proof of Theorem 3.

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