A Product Inequality for Extreme Distances*

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Abstract

Let $p_1, \ldots, p_n$ be n distinct points in the plane, and assume that the minimum inter-point distance occurs $s_{\min}$ times, while the maximum inter-point distance occurs $s_{\max}$ times. It is shown that $s_{\min} s_{\max} \leq \frac{9}{8}n^2 + O(n)$; this settles a conjecture of Erdős and Pach (1990).

Keywords: extreme distances; repeated distances.

1 Introduction

Let $p_1, \ldots, p_n$ be n distinct points in the plane, and assume that the minimum inter-point distance occurs $s_{\min}$ times, while the maximum inter-point distance occurs $s_{\max}$ times. It is well-known that $s_{\min} \leq 3n$ and $s_{\max} \leq n$; see, e.g., [3, Ch. 13]; and these classical bounds immediately imply that $s_{\min} s_{\max} \leq 3n^2$. Erdős and Pach [1] asked for a proof or disproof of the following sharper product inequality:

$$s_{\min} s_{\max} \leq \frac{9}{8}n^2 + o(n^2).$$

The authors also remarked that this inequality, if true, essentially cannot be improved; and this would follow from a construction of E. Makai Jr. (not discussed in their paper). Indeed, the main term in the inequality cannot be improved: the point configuration exhibited in Fig. 1 has $s_{\min} = \frac{3}{4}n + \frac{3}{4}n - O(\sqrt{n}) = \frac{3}{2}n - O(\sqrt{n})$, and $s_{\max} = \frac{3}{4}n$ (provided that the circular arc subtends an angle of $60°$), and so $s_{\min} s_{\max} = \frac{9}{8}n^2 - O(n\sqrt{n})$. The $m = \frac{1}{4}n$ interior points make a section of a unit triangular lattice with $\lfloor 3m - \sqrt{12m-3} \rfloor$ unit distances, where the minimum inter-point distance is equal to 1; see [2] or [3, p. 211]. For the $\frac{3}{4}n - 1$ boundary points on the circular arc, successive points are at distance 1.

Here we prove the claimed inequality in a slightly stronger form (with a linear lower order term).

Theorem 1. *Let $p_1, \ldots, p_n$ be n distinct points in the plane, and let $s_{\min}$ and $s_{\max}$ denote the multiplicity of the minimum and maximum inter-point distance, respectively. Then $s_{\min} s_{\max} \leq \frac{9}{8}n^2 + O(n)$.*

Definitions and notations. For a point set S, $\mathrm{conv}(S)$ denotes the convex hull of S, and $\partial\mathrm{conv}(S)$ denotes the boundary of $\mathrm{conv}(S)$. The perimeter of a polygon P is denoted by $\mathrm{per}(P)$. A convex polygon is one in *strictly convex* position, i.e., no three boundary points are collinear.

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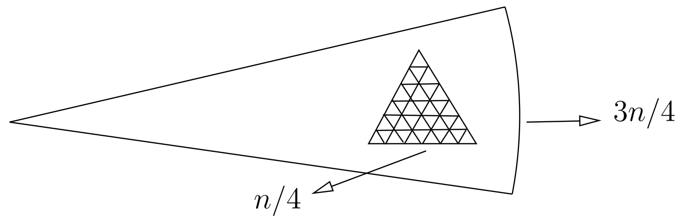

Figure 1: An n-element point set with $\frac{3}{4}n$ points on the convex hull and $\frac{1}{4}n$ interior points. $\frac{3}{4}n-1$ boundary points are evenly distributed on a circular arc of radius $\Theta(\sqrt{n})$ centered at the leftmost point.

Preliminaries. Let $S = \{p_1, \ldots, p_n\}$ be a set of n distinct points in the plane. Given two points p and q, let $\ell(p, q)$ denote the line determined by p and q. Let δ and Δ denote the minimum and maximum pairwise distance of S, respectively. We may assume that $\delta = 1$; a standard packing argument then yields $\Delta = \Omega(\sqrt{n})$. Let G_δ and G_Δ denote the respective graphs. As mentioned earlier, we have $|E(G_\delta)| \leq 3n$ and $|E(G_\Delta)| \leq n$.

Throughout the proof, graph adjacency refers to G_δ, unless specified otherwise. For any point $u \in S$, let $\deg(u)$ denote its degree in G_δ; it is well known that $\deg(u) \leq 6$ for any $u \in S$. For any point $u \in S$, let $\Gamma(u) = \{v \in S \ : \ uv \in E(G_\delta)\}$; i.e., $\Gamma(u)$ is the set of vertices adjacent to u in G_δ. For a point u, let $x(u)$ and $y(u)$ denote its x- and y-coordinates respectively.

2 Setup of the proof

Let $H \subseteq S$ denote the set of (extreme) vertices of $\mathrm{conv}(S)$ labeled in a clockwise manner: $H = \{u_1, u_2, \ldots, u_h\}$, and so that indices can be read in a circular fashion, i.e., $u_{h+1} = u_1$. We say that a vertex $u_i \in H$ has a *flat neighborhood* if the interior angles of the seven vertices $\{u_{i-3}, u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}, u_{i+3}\}$ all belong to the interval $(179^\circ, 180^\circ)$. Observe that the number of vertices of $\mathrm{conv}(S)$ that do *not* have flat neighborhoods is $O(1)$.

Let $F \subseteq H$ denote the set of vertices of $\mathrm{conv}(S)$ that have flat neighborhoods. Let $D \subseteq H$ denote the set of vertices of $\mathrm{conv}(S)$ that are endpoints of some diameter pair. Put $|D| = d$, $f = |F|$, and recall that $h = |H|$; as such, $d \leq h$ and $f \leq h$.

The set of points S can be partitioned into three parts as $S = H \cup H' \cup I$, where

- H is the set of extreme vertices of $\mathrm{conv}(S)$; an element of H can be in any of the following sets $D \cap F$, $D \setminus F$, $F \setminus D$, or $S \setminus (D \cup F)$. Let $u_1, \ldots, u_h$ (where $u_{h+1} = u_1$) be the extreme vertices of $\mathrm{conv}(S)$ in clockwise order.
- H' is the set of points on $\partial\mathrm{conv}(S)$ that are not in H (the interior angle of each vertex in H' is 180°).
- I is the set of interior vertices, i.e., those that are not on $\partial\mathrm{conv}(S)$.

As mentioned earlier [3, Ch. 13], we have

$$s_{\max} \leq d \leq h. \tag{1}$$

Indeed, the endpoints of any diameter pair must be extreme points on the boundary of $\mathrm{conv}(S)$. If $d \leq n/3$, then $s_{\max} \leq d \leq n/3$ and consequently, $s_{\min} s_{\max} \leq 3n \cdot \frac{1}{3}n = n^2$, as required (with room to spare). We therefore subsequently assume that $d \geq n/3$; and so we have $h \geq d \geq n/3$.

Lemma 1. *If $h \geq n/3$, then $\Delta \geq \frac{n}{3\pi}$; in particular $\Delta = \Omega(n)$.*

Proof. Let $p = \mathrm{per}(\mathrm{conv}(S))$; since $\delta = 1$ and $h \geq n/3$, we have $p \geq n/3$. By a standard isoperimetric inequality, $p \leq \pi\Delta$; see, e.g., [4]. Putting the two inequalities together yields $\Delta \geq \frac{n}{3\pi}$, as required. □

3 Charging scheme

Assume that each point in S carries an initial charge equal to its degree in G_δ (at most 6). The scheme we discuss below transfers a unit charge from each convex hull vertex of degree 3 that belongs to $D \cap F$ to one or two interior vertices, in such a way that the final charge of each interior vertex is at most 6. This achieves the desired effect that the endpoints on the convex hull of these edges are left with a charge of 2 (while their initial charge was 3). Once this goal is achieved, the upper bound we need on the number of unit distances will follow from Lemma 4, as detailed in Section 5.

The main difficulties posed by this plan are (i) deciding how to implement the charging scheme; and (ii) verifying its validity (namely that the final charge of each interior vertex is bounded from above by 6). We next describe the charging scheme, after which we show in Lemma 3 that it works as intended.

Overview. Recall that $u_1, \ldots, u_h$ (where $u_{h+1} = u_1$) are the extreme vertices of $\mathrm{conv}(S)$ in clockwise order. For any extreme vertex with a flat neighborhood $u_i \in F$, let Σ_{u_i} be an orthogonal coordinate system whose origin is u_i, and where the x-axis is a supporting line of $\mathrm{conv}(S)$ incident to u_i, and S lies in the closed halfplane below the x-axis. See Fig. 2. More precisely: if $u_i u_{i+1} \in G_\delta$ and there exists $v \in I$ s.t. $u_i v, u_{i+1} v \in G_\delta$ (i.e., $\Delta u_i u_{i+1} v$ is an equilateral triangle), the x-axis will be chosen as $\overrightarrow{u_i u_{i+1}}$; otherwise, the x-axis will be chosen so that $S \setminus \{u_i\}$ lies strictly below this line and the bisector of the interior angle $\angle u_{i-1} u_i u_{i+1}$ is the negative direction of the y-axis.

Having defined Σ_{u_i}, consider the rectangle $R_{u_i} = [x(u_i) - 7/4, x(u_i) + 7/4] \times [y(u_i) - 2, y(u_i)]$ in this system. When sending charge from u_i, a reference will be made to R_{u_i} (in the details of the charging scheme).

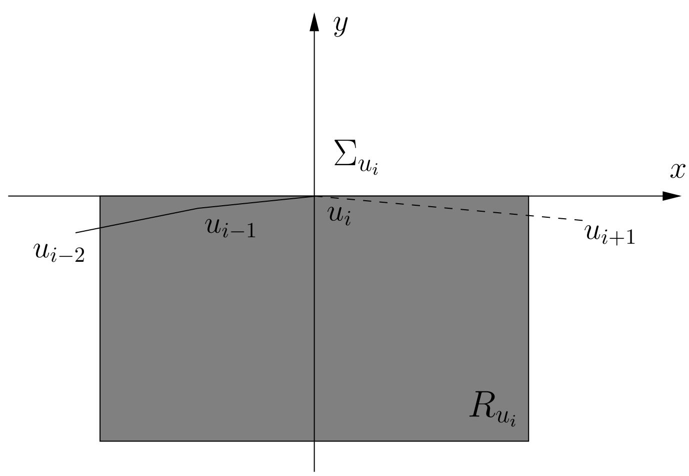

Figure 2: The coordinate system Σ_{u_i} and the axis-aligned rectangle R_{u_i} for a vertex $u_i \in H$ with a flat neighborhood.

Vertices in $H = \{u_1, \ldots, u_h\}$ are processed one by one in this order (pairs of adjacent vertices of H corresponding to equilateral triangles in G_δ are processed at the same time). Equivalently, we keep the coordinate system fixed (with the two axes horizontal and vertical) and rotate S counterclockwise so that u_i is the highest vertex in S at the time it is processed; see Fig. 3 (middle) for the case when the x-axis will be chosen as $\overrightarrow{u_i u_{i+1}}$; and see Fig. 3 (left) for the remaining case. In either case, the point set S is contained in the closed halfplane below the x-axis.

Let $u_i \in D \cap F$ be an extreme vertex of degree 3; if the vertices in $\Gamma(u_i)$ are ordered from left to right, let $v_i \in \Gamma(u_i)$ be the second (middle) element. We refer to the edge $u_i v_i$ as the *middle edge*[1] associated (and incident) to u_i. It is convenient to think about the unit charge that gets transferred from a vertex in $D \cap F$ of degree 3 as being associated to the middle edge incident to that vertex.

It is worth examining the configuration in Fig. 3 (right). Let $u_i, u_{i+1} \in D \cap F$ be extreme vertices of degree 3; if $u_i v$ and $u_{i+1} v$ are unit edges incident to v connecting v with two non-adjacent extreme vertices u_i and u_{i+1} (i.e., $|u_i u_{i+1}| > 1$), then $u_i v_i$ and $u_{i+1} v_{i+1}$ are the middle edges whose charge is transferred to interior vertices.

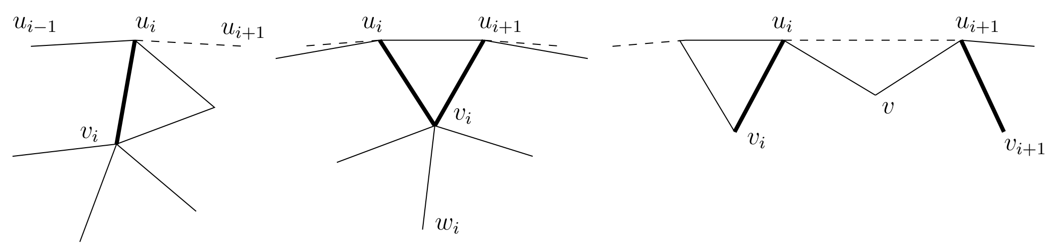

Figure 3: Illustrations to the scenarios described in the text. Edges in G_δ are drawn in solid, and middle edges are drawn in bold.

Charging rules. When handling the current vertex u_i, or two consecutive vertices u_i, u_{i+1} that belong to a unit equilateral triangle, we use the coordinate system Σ_{u_i}. Let $u_i v_i$ be the middle edge from u_i, where v_i is an interior vertex. We distinguish four cases, depending on whether (i) the degree of v_i is 6 or less than 6; and (ii) v_i is connected to one or two vertices on $\partial\text{conv}(S)$. The following charging rules are observed:

1. Every middle edge has its unit charge distributed to one or two interior vertices.
2. Charging amounts can be 1/2 or 1: we sometimes transfer the entire unit charge of a middle edge to an interior vertex and sometimes split the unit charge into two equal parts, each being 1/2, that are sent to two different interior vertices.
3. The unit charge on the middle edge incident to u_i is distributed to one or two interior vertices at distance at most 2 from u_i in G_δ; i.e., this charging process can only affect vertices in $\Gamma(u_i) \cup \Gamma(\Gamma(u_i))$.
4. The property that charges are only distributed to interior vertices is a consequence of the fact that such charges originate at vertices in $D \cap F$ and $\Delta \geq \frac{n}{3\pi}$ by Lemma 1 (and so the opposite boundary is far away).

Before the execution of the charging scheme, we clearly have $s_{\min} = \frac{1}{2} \sum_{p \in S} \deg(p)$. The charging scheme that is put in place transfers one unit from each extreme vertex of degree 3 that is an element of $D \cap F$ to one or two interior vertices. After completion, $s_{\min}$ can be calculated in an alternative way, as half the sum of final charges of all vertices.

[1]In all subsequent figures, unit edges are drawn in solid and middle edges are drawn in bold.

Details. *Case 1:* $\deg(v_i) = 6$, and $u_i v_i$ is the unique unit edge incident to v_i connecting v_i with an extreme vertex; see Fig. 4. Note that the six vertices in $\Gamma(v_i)$ form a regular hexagon of unit side-length. Let $a, b \in \Gamma(u_i) \cap \Gamma(v_i)$ be the two common neighbors of u_i and v_i on the left and right, respectively. Note that $\deg(a) \leq 5$, and similarly, $\deg(b) \leq 5$; indeed, if $\deg(a) = 6$ (or $\deg(b) = 6$), one element in $\Gamma(a)$ (resp., $\Gamma(b)$) would lie strictly above u_i, a contradiction. Distribute the unit charge on edge $u_i v_i$ into two equal parts: $1/2$ to the left interior vertex a and $1/2$ to the right interior vertex b. Observe that $a, b \in R_{u_i}$.

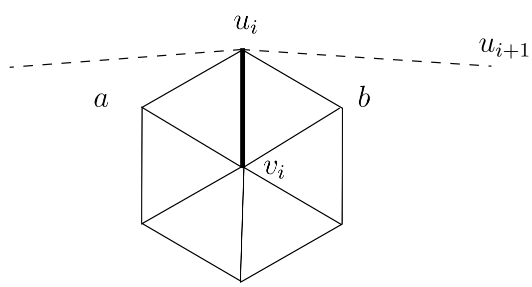

Figure 4: Case 1. All points lie in the closed halfplane below the horizontal line incident to u_i.

Case 2: $\deg(v_i) = 6$, where $u_i v_i$ and $u_{i+1} v_i$ are unit edges incident to v_i, connecting v_i with extreme vertices u_i and u_{i+1}, necessarily adjacent in G_δ; see Fig. 5 (left). The argument assumes that both u_i and u_{i+1} are elements of $D \cap F$, since otherwise, there is no need to transfer charge from the respective unit edges. Note that the six vertices in $\Gamma(v_i)$ form a regular hexagon of unit side-length. Let $a \in \Gamma(u_i) \cap \Gamma(v_i)$ be the interior vertex on the left, and $b \in \Gamma(u_{i+1}) \cap \Gamma(v_i)$ be the interior vertex on the right. Note that $\deg(a) \leq 5$ and $\deg(b) \leq 5$; indeed, if say, $\deg(a) = 6$ (or $\deg(b) = 6$), the interior angle at u_i (resp., at u_{i+1}) would be $180°$, a contradiction, since we have assumed that $u_i, u_{i+1} \in D$.

We further identify other vertices of low degree that will be charged. Let $w_i, w_{i+1} \in \Gamma(v_i)$ be the two neighbors of v_i below it, as in Fig. 5 (right). Our charging scheme is symmetric, and here we show how to distribute the unit charge on edge $u_{i+1} v_i$ to vertex b and some other interior vertex (the distribution of the unit charge of edge $u_i v_i$ is analogous, involving vertex a and some other interior vertex).

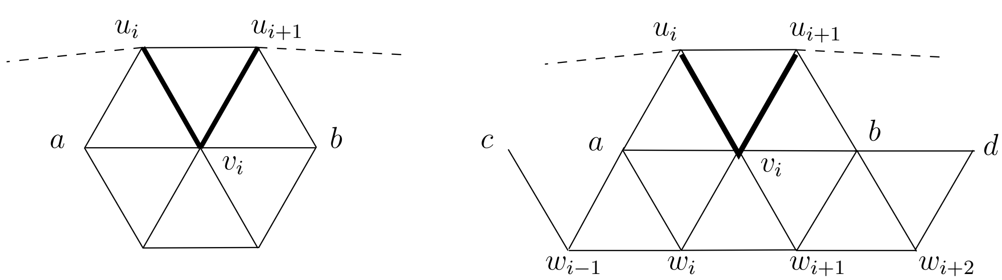

Figure 5: Case 2. All points lie in the closed halfplane below the horizontal line $\ell(u_i, u_{i+1})$.

If $\deg(w_{i+1}) \leq 5$, distribute the unit charge on edge $u_{i+1} v_i$ into two equal parts: $1/2$ to interior vertex b and $1/2$ to the interior vertex w_{i+1}. We subsequently assume that $\deg(w_{i+1}) = 6$. Let $w_{i+2} \in \Gamma(b) \cap \Gamma(w_{i+1})$ be the interior vertex on the line $\ell(w_i, w_{i+1})$ to the right. If $\deg(w_{i+2}) \leq 5$, distribute the unit charge on edge $u_{i+1} v_i$ into two equal parts: $1/2$ to interior vertex b and $1/2$ to the interior vertex w_{i+2}. We subsequently assume that $\deg(w_{i+2}) = 6$. Let $d \in \Gamma(b) \cap \Gamma(w_{i+2})$ be the interior vertex on the line $\ell(v_i, b)$ to the right. Observe that $\deg(d) \le 4$: since each element of $\Gamma(d) \setminus \{b, w_{i+2}\}$ must lie strictly below the line $\ell(w_{i+2}, d)$, there are at most two such vertices. In this last case, distribute the unit charge on edge $u_{i+1}v_i$ into two equal parts: 1/2 to the interior vertex b and 1/2 to the interior vertex d. Observe that $b, d, w_{i+1}, w_{i+2} \in R_{u_{i+1}}$, and similarly that $a, c, w_i, w_{i-1} \in R_{u_i}$.

Case 3: $\deg(v_i) \le 5$, and u_iv_i is the unique unit edge incident to v_i connecting v_i with an extreme vertex. If $\deg(v_i) \le 4$, charge edge u_iv_i to the interior vertex v_i; i.e., v_i receives a unit charge; see Fig. 6 (left).

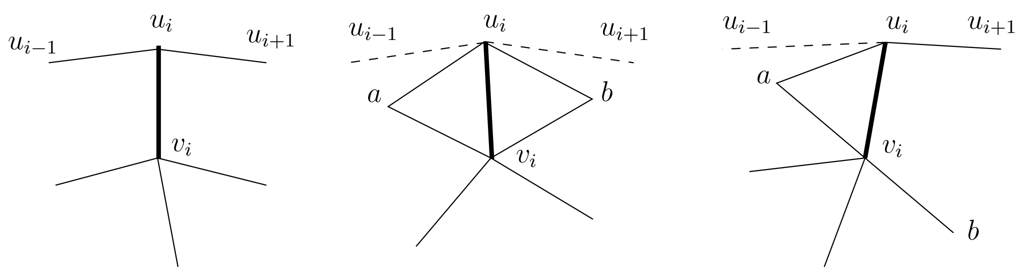

Figure 6: Case 3. Left: $\deg(v_i) = 4$. Middle: b is the highest among $\{a, b\}$. Right: a is the highest among $\{a, b\}$. All points lie in the closed halfplane below the horizontal line incident to u_i.

If $\deg(v_i) = 5$, refer to Fig. 6 (middle and right). Let a and b be the two neighbors of v_i left and right of the edge u_iv_i, respectively, and $h(a, b)$ denote the element of $\{a, b\}$ which is the highest i.e., closest to the x-axis of Σ_{u_i} (with ties broken arbitrarily). Since the open halfplane below the horizontal line incident to v_i contains at most three elements of $\Gamma(v_i)$, we have $y(h(a, b)) \ge y(v_i)$. Since $\deg(u_i) = 3$ and $\delta = 1$ is the minimum distance, $h(a, b)u_i$ is an edge in G_δ and so the triangle $\Delta h(a, b)u_iv_i$ is equilateral. This further implies that $h(a, b)$ has degree at most 5; since otherwise, the y-coordinate of one of its neighbors (with respect to Σ_{u_i}) would be non-negative, a contradiction. Distribute the unit charge on edge u_iv_i into two equal parts: 1/2 unit to v_i and 1/2 unit to $h(a, b)$. Observe that $v_i, a, b \in R_{u_i}$.

Case 4: $\deg(v_i) \le 5$, where u_iv_i and $u_{i+1}v_i$ are unit edges incident to v_i, connecting v_i with two extreme vertices u_i and u_{i+1}. Since u_iv_i and $u_{i+1}v_i$ are middle edges and $u_i, u_{i+1} \in F$, these vertices are necessarily adjacent in G_δ; see Fig. 7. If $\deg(v_i) \le 4$, distribute the two units of charge on u_iv_i and $u_{i+1}v_i$ to v_i. Assume now that $\deg(v_i) = 5$ and let w_i denote a vertex in $\Gamma(v_i)$ that is the lowest, i.e., farthest from $\ell(u_iu_{i+1})$ (with ties broken arbitrarily).

If $\deg(w_i) \le 5$, distribute the two units of charge for edges u_iv_i and $u_{i+1}v_i$ into two equal parts: one unit to v_i and one unit to w_i; see Fig. 7 (left). Observe that $v_i, w_i \in R_{u_i}$

Assume now that $\deg(w_i) = 6$; observe that the six vertices in $\Gamma(w_i)$ form a regular hexagon of unit side-length. Let $a, b \in \Gamma(v_i) \cap \Gamma(w_i)$ be the two common neighbors of v_i and w_i on the left and respectively the right of the edge w_iv_i; see Fig. 7 (right). We claim that $\deg(a) \le 5$ and $\deg(b) \le 5$. We may assume that $\angle av_iu_i \ge 90^\circ \ge \angle bv_iu_{i+1}$.

If $\deg(a) = 6$, let v_{i-1} be the next counterclockwise vertex after v_i in $\Gamma(a)$. Since the triangle $\Delta av_{i-1}v_i$ is equilateral and $\angle av_iu_i \ge 90^\circ$, it follows that $v_{i-1}v_i$ is yet another edge in G_δ, contradicting the assumption that $\deg(v_i) = 5$.

If $\deg(b) = 6$, then bu_{i+1} is an edge in G_δ, thus $v_ib \parallel u_iu_{i+1}$ and so v_ib is horizontal. Let c be the next clockwise vertex after u_{i+1} in $\Gamma(b)$. Then $u_{i+1}c$ is also horizontal, thus $c \in \partial\mathrm{conv}(S)$, which implies that the interior angle at u_{i+1} is 180°, which is a contradiction (we have assumed that $u_i, u_{i+1} \in D$).

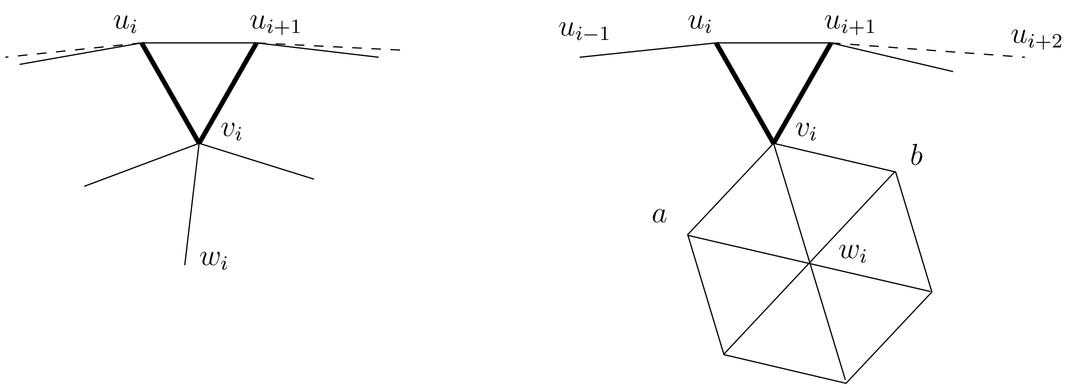

Figure 7: Case 4. Left: $\deg(w_i) = 5$. Right: $\deg(w_i) = 6$. All points lie in the closed halfplane below the horizontal line $\ell(u_i, u_{i+1})$.

Since each of the two assumptions $\deg(a) = 6$ and $\deg(b) = 6$ leads to a contradiction, this proves the claim. Distribute the two unit charges for edges u_iv_i and $u_{i+1}v_i$ as one unit to v_i, 1/2 unit to a and 1/2 unit to b. (This can be also viewed as distributing the unit charge for u_iv_i as 1/2 unit to v_i and 1/2 unit to a, and distributing the unit charge for $u_{i+1}v_i$ as 1/2 unit to v_i and 1/2 unit to b.) Observe that $v_i, a \in R_{u_i}$ and $v_i, b \in R_{u_{i+1}}$.

Illustration. An example illustrating the final charges in a few representative cases is shown in Fig. 8.

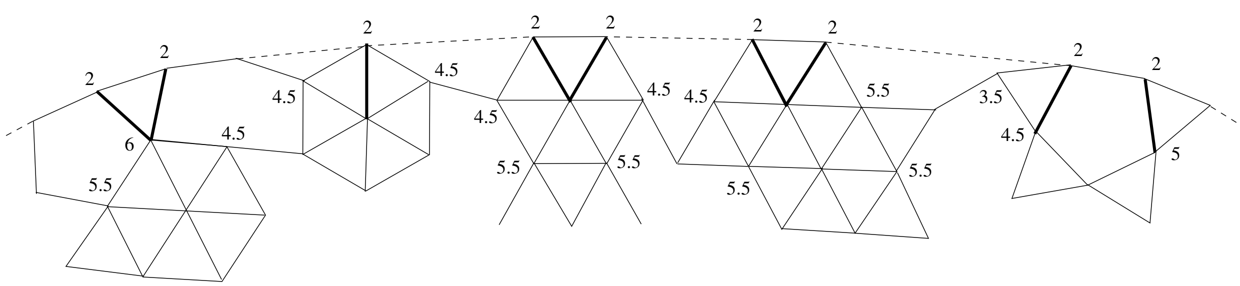

Figure 8: Charging illustrations for vertices on the upper hull; middle edges adjacent to extreme vertices of degree 3 in $D \cap F$ are drawn in thick lines.

4 Charging scheme analysis

By direct inspection of the scheme we note the two properties announced prior to describing the charging scheme:

Observation 1. *The following hold:* (i) *Unit charges associated to middle edges are distributed to interior vertices in amounts of* 1/2 *or* 1. (ii) *The unit charge on the middle edge incident to u_i is distributed to one or two interior vertices at distance at most* 2 *in G_δ; i.e., this process can only affect vertices in $\Gamma(u_i) \cup \Gamma(\Gamma(u_i))$.*

The following lemma specifies the range affected by one charge distribution.

Lemma 2. *Let $u_i \in D \cap F$ be a vertex of degree 3 that sends charge to some interior vertex $v \in \Gamma(u_i) \cup \Gamma(\Gamma(u_i))$, where v is not necessarily unique. Then v can only receive charges from elements of $\{u_{i-3}, u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}, u_{i+3}\}$.*

Proof. Write $u = u_i$ (for short). Consider the coordinate system Σ_u, and the rectangle $R_u = [x(u) - 7/4, x(u) + 7/4] \times [y(u) - 2, y(u)]$ in this system. By the charging scheme, u can only send charges to interior vertices contained in R_u. Consider the larger rectangle $R'_u = [x(u) - 15/4, x(u) + 15/4] \times [y(u) - 4, y(u)] \supset R_u$; refer to Fig 9. Since v can only receive charges from vertices at distance at most 2 from it, any element sending charges to v would be contained in R'_u.

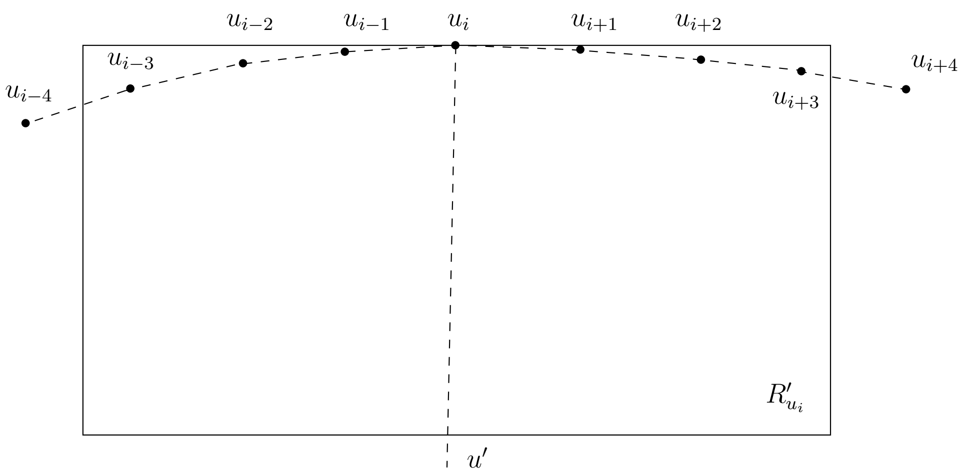

Figure 9: The flat neighborhood of $u = u_i$ and a diameter pair (u, u'). All points lie in the closed halfplane below the horizontal line incident to u_i.

Since $u \in D \cap F$, u is an endpoint of a diameter pair, say, uu', where $u' \in D$. Observe that the ray $\overrightarrow{uu'}$ makes an angle of at most $1°$ with $\overrightarrow{r_u}$, the vertical ray from u pointing downwards. Indeed, otherwise one of the two distances $|u_{i-1}u'|$ and $|u_{i+1}u'|$ would be larger than $|uu'| = \Delta$, as the longest side in an obtuse triangle. Recall that $\Delta = \Omega(n)$ by Lemma 1; we may assume that n is large enough, e.g., $n \geq 100$, so that $\Delta \geq 10$. By convexity, $\text{conv}(u_{i-3}, u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}, u_{i+3}, u')$ is empty of points from H in its interior. Recall that u has a flat neighborhood, and intuitively, this implies that v cannot receive charges from the 'other side' of the boundary. More precisely, since u has a flat neighborhood, the rectangle R'_u does not contain any elements of $H \setminus \{u_{i-3}, u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}, u_{i+3}\}$, and so v cannot receive charges from elements in $H \setminus \{u_{i-3}, u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}, u_{i+3}\}$, as required. □

We next formulate and prove the main property accomplished by the charge distribution.

Lemma 3. *The final charge for any interior vertex is at most 6. Moreover, any interior vertex may receive charges from at most two vertices in $D \cap F$.*

Proof. By Lemma 2 it suffices to bound from above the charge received by an interior vertex from nearby vertices in $D \cap F$. Any interior vertex receiving charge from the left can be uniquely associated with the corresponding distribution case in the charging scheme; and similarly, any interior vertex receiving charge from the right can be uniquely associated with the corresponding distribution case in the charging scheme. As such, any interior vertex may receive charges from at most two vertices in $D \cap F$.

Overcharging by Case 1 only: Let v be an interior vertex of degree 5 that is charged as b in Fig. 4 from the left (i.e., from an edge u_iv_i on the left), and as a in Fig. 4 from the right (i.e., from an edge u_iv_i on the right). The charges received sum up to at most $\frac{1}{2} + \frac{1}{2} = 1$, as required.

Overcharging by Case 1 and Case 2: The argument is similar to that for the previous case. Let v be an interior vertex that is charged from the left as b or d in Fig. 5 (right) and is charged from the right as a in Fig. 4. The charges received sum up to at most $\frac{1}{2} + \frac{1}{2} = 1$, as required.

Overcharging by Case 1 and Case 3: Assume that an interior vertex receives a unit charge as vertex v_i in Fig. 6 according to Case 3. This happens only when $\deg(v_i) \le 4$; and then it is easy to see that no overcharging can occur even if v_i receives 1/2 unit from the left and from the right (according to Case 1). In the remaining case, $\deg(v_i) = 5$, both vertices that get charged according to Case 3, only receive 1/2 unit charge each. Since charges received from Case 1 are limited to 1/2 unit, the charges received by v_i sum up to at most $\frac{1}{2} + \frac{1}{2} = 1$, as required.

Overcharging by Case 1 and Case 4: Assume that some interior vertex receives a unit charge from the left according to Case 4 and 1/2 unit charge from the right according to Case 1. See Fig. 10 and Fig. 11, and observe that $j = i + 2$. First, consider the situation in Fig. 10, when $\deg(w_i) = 5$. If the overcharged vertex is $v_i = c$, the interior vertex c would be adjacent to three extreme vertices (u_i, u_{i+1}, u_{i+2}), a contradiction. If the overcharged vertex is $w_i = c$, the distance from w_i to $\ell(u_iu_{i+1})$ is at least $2\frac{\sqrt{3}}{2} = \sqrt{3}$; on the other hand, the distance from c to the same line is less than 1, since u_{i+1} has a flat neighborhood. Therefore such an overcharging cannot occur.

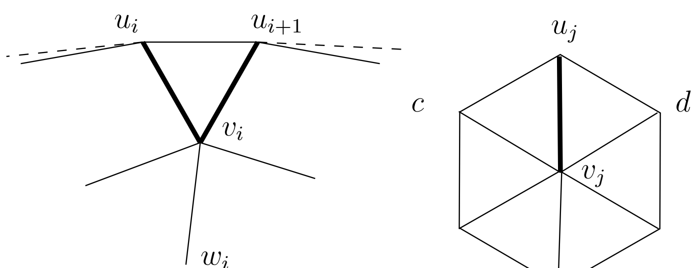

Figure 10: Overcharging by Case 4 (left) and Case 1 (right); $\deg(w_i) = 5$.

Second, consider the situation in Fig. 11, when $\deg(w_i) = 6$. If the overcharged vertex is $b = c$, the charges received sum up to at most $\frac{1}{2} + \frac{1}{2} = 1$, as required.

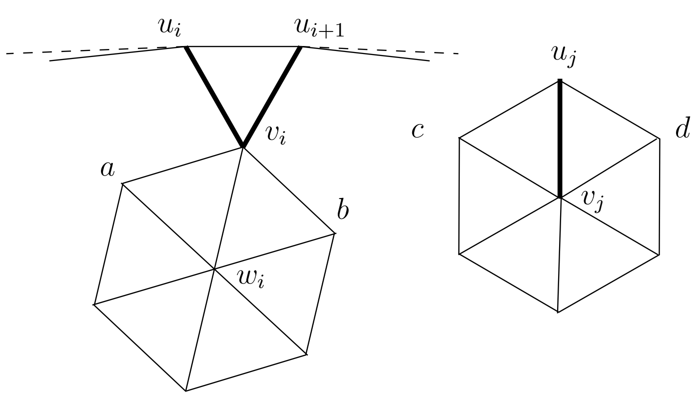

Figure 11: Overcharging by Case 4 (left) and Case 1 (right); $\deg(w_i) = 6$.

Overcharging by Case 2 only: Each charge received by an interior vertex is Case 2 is equal to 1/2; as such, an interior vertex of degree at most 5 can receive at most 1/2 units from the left and at most 1/2 units from the right, as required.

Overcharging by Case 2 and Case 3: Refer to Fig. 12. Assume for contradiction that a vertex of degree at most 5 receives a 1/2 unit charge from the left according to Case 2 (as vertex b, w_{i+1}, w_{i+2}, or d), and a 1/2 unit charge as vertex c or v_j according to Case 3. It is clear that $j \geq i+2$. The charge received is at most $\frac{1}{2} + \frac{1}{2} = 1$, as required. The situation when $\deg(v_j) \leq 4$ is similarly easy to analyze.

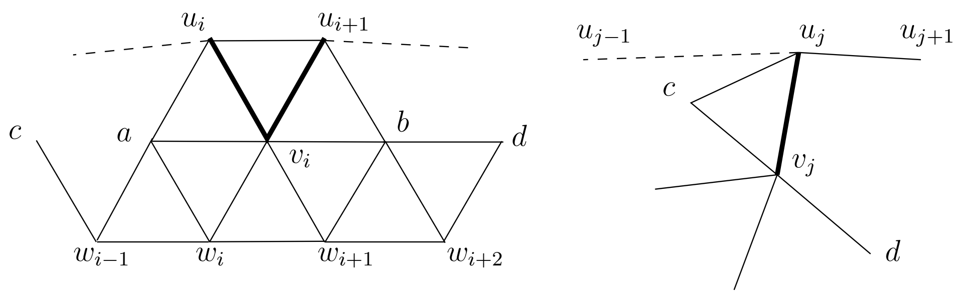

Figure 12: Overcharging by Case 2 (left) and Case 3 (right).

Overcharging by Case 2 and Case 4: Assume for contradiction that an interior vertex has degree 5 and receives a 1/2 unit charge from the left according to Case 2, and a one unit charge from the right according to Case 4; see Fig. 13. Then

- vertex w_{i+2} on the left must coincide with vertex w_j on the right.
- vertex b on the left must coincide with vertex e on the right.
- vertex d on the left must coincide with vertex v_j on the right.

However this is impossible to achieve with $u_i, u_{i+1}, u_j, u_{j+1}$ consecutive extreme vertices with flat neighborhoods, since u_j, u_{j+1} would need to lie strictly below $\ell(u_i, u_{i+1})$, while the triangle $\Delta v_j u_j u_{j+1}$ is equilateral and $\ell(u_j, u_{j+1})$ is almost horizontal (recall from Case 2 that $\deg(d) \leq 4$ due to some restrictions imposed on its neighbors). The above conditions imply that v_j is lower than d and so the two points cannot coincide.

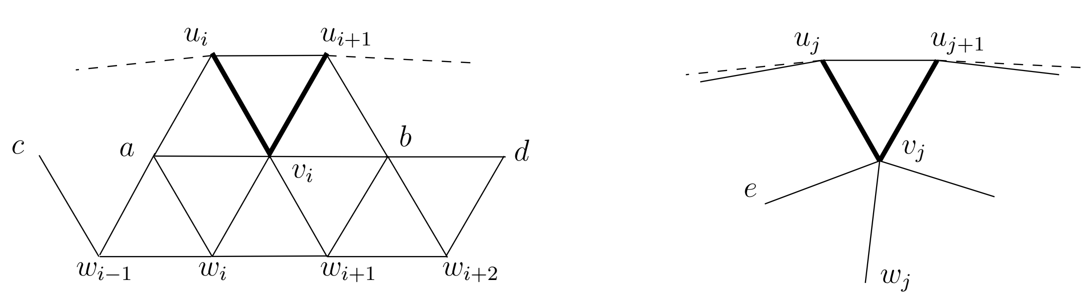

Figure 13: Overcharging by Case 2 (left) and Case 4 (right).

Overcharging by Case 3 only: Since v_i is adjacent to exactly one extreme vertex, v_i cannot receive multiple charges. Any other vertex can only receive a 1/2 unit charge from the left and a

1/2 unit charge from the right. As such, the charge received is bounded from above by $\frac{1}{2} + \frac{1}{2} = 1$, as required.

Overcharging by Case 3 and Case 4: Observe that vertex v_i in Case 4, see Fig. 7 (left or right), is adjacent to exactly two extreme vertices; consequently it cannot receive any charge according to the procedure in Case 3. Similarly, observe that vertex w_i in Case 4, see Fig. 7 (left), is not adjacent to any extreme vertex; consequently it cannot receive any charge according to the procedure in Case 3. Consider now the scenario illustrated in Fig. 14 in which $\deg(v_i) = \deg(v_j) = 5$; observe

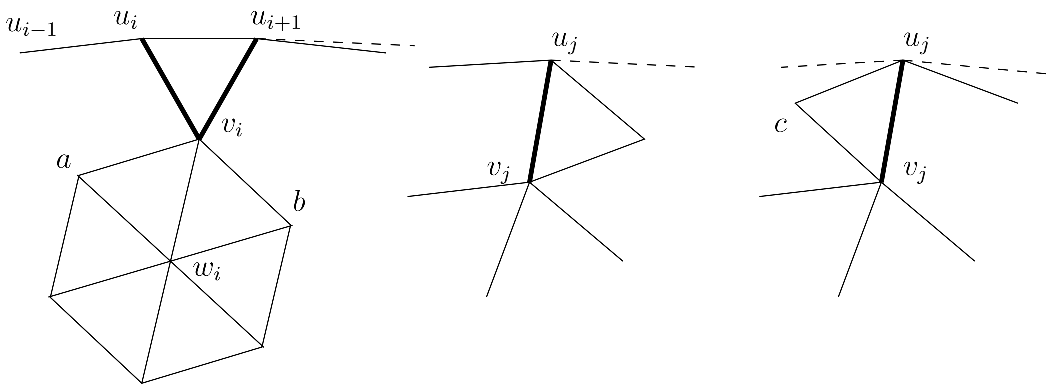

Figure 14: Overcharging by Case 4 (left) and Case 3 (middle and right).

that $j = i + 2$. If $b = v_j$ receives a 1/2 unit charge according to Case 4 and another 1/2 unit charge according to Case 3, the charge received is bounded from above by $\frac{1}{2} + \frac{1}{2} = 1$, as required; see Fig. 14 (middle). Similarly, if $b = c$ receives a 1/2 unit charge according to Case 4 and another 1/2 unit charge according to Case 3, the charge received is at most $\frac{1}{2} + \frac{1}{2} = 1$, as required; see Fig. 14 (right). Therefore such an overcharging cannot occur.

Overcharging by Case 4 only: Refer to Fig. 15 and Fig. 16; observe that $j = i+2$. One possibility is having $w_i = a$, with w_i receiving a unit charge according to Case 4 (on the left) and a receiving a 1/2 unit charge according to the same case (on the right); see Fig. 15. However this requires $v_i v_j$ to be yet another edge in G_δ beyond the five incident to v_j, contradicting the assumption of Case 4 that $\deg(v_j) = 5$.

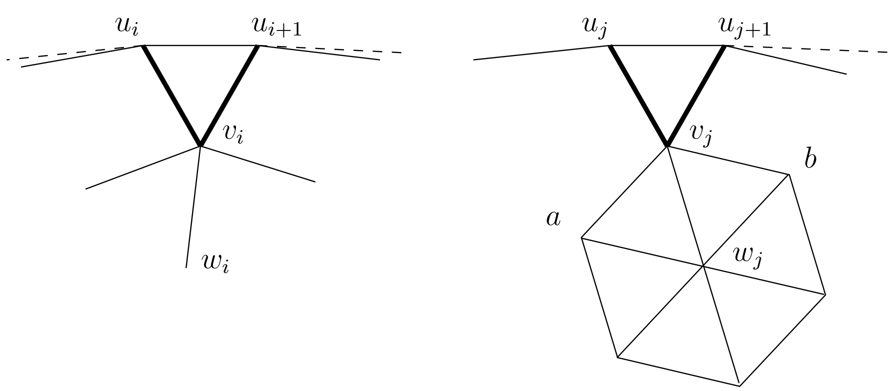

Figure 15: Overcharging by Case 4 only.

Another possibility is having $b = c$, with b receiving a 1/2 unit charge according to Case 4 (on the left) and c receiving a 1/2 unit charge according to the same case (on the right); see Fig. 16.

The charge received is at most $\frac{1}{2} + \frac{1}{2} = 1$, as required.

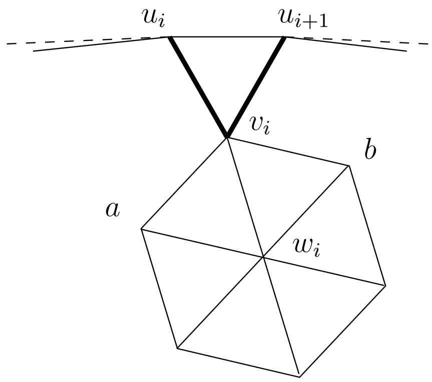

Figure 16: Overcharging by Case 4 only.

With all possibilities of potential overcharging having been analyzed, the proof of Lemma 3 is now complete. □

5 Conclusion of the proof

We can now finalize the proof of Theorem 1.

Lemma 4. $s_{\min} \leq 3n - 2d + O(1)$.

Proof. Assume that each element of I carries an initial charge equal to its degree in G_δ (at most 6). Note that the degree of each element of H is at most 3; indeed, if $\deg(u) = 4$, then the interior angle at u equals 180°, and so u is not an extreme vertex of $\mathrm{conv}(S)$. In particular, each element of $D \cap F$ has degree at most 3.

Each vertex of $D \cap F$ of degree 3 in G_δ sends (distributes) a unit charge to one or two interior vertices of degree at most 5; so that the final charge of each interior vertex is at most 6. Each vertex receives a charge at most 2 and the final charge of each element of $D \cap F$ is 2. All these are consequences of Lemma 3. A key observation is that $|F \cap D| \geq |D| - O(1)$, since there are only $O(1)$ elements of D that do not have flat neighborhoods. Assuming the charging procedure finalized, we have

$$\begin{aligned} 2s_{\min} &= \sum_{p \in S} \deg(p) \leq 3|H \setminus F \cap D| + 2|F \cap D| + 6|S \setminus H| \\ &= 3h - 3|F \cap D| + 2|F \cap D| + 6n - 6h \\ &= 6n - 3h - |F \cap D| \leq 6n - 3d - d + O(1) \\ &= 6n - 4d + O(1), \end{aligned}$$

as required. □

Proof of Theorem 1. Using the inequalities on $s_{\min}$ and $s_{\max}$ stated in Lemma 4 and Equation (1), respectively, we obtain

$$s_{\min} s_{\max} \leq (3n - 2d + O(1))\, d \leq \frac{9}{8} n^2 + O(n),$$

as required. Indeed, setting $x = d/n$ yields the quadratic function $f(x) = x(3 - 2x)$, which attains its maximum value $\frac{9}{8}$ for $x = \frac{3}{4}$. Thus $(3n - 2d)d \leq \frac{9}{8}n^2$ and we also have $O(1)d = O(d) = O(n)$; adding these two inequalities yields the one claimed above. This concludes the proof of Theorem 1.

It is crucial that only vertices with degree 3 in $D \cap F$ are charged out. It is possible to exhibit sections (or subsets) of the regular hexagonal lattice where it is not possible to charge the diameter endpoints to interior vertices, however, none of these have flat neighborhoods.

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