Monotone Paths in Geometric Triangulations*

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Abstract

(I) We prove that the (maximum) number of monotone paths in a geometric triangulation of n points in the plane is $O(1.7864^n)$. This improves an earlier upper bound of $O(1.8393^n)$; the current best lower bound is $\Omega(1.7003^n)$.

(II) Given a planar geometric graph G with n vertices, we show that the number of monotone paths in G can be computed in $O(n^2)$ time.

Keywords: monotone path, triangulation, counting algorithm.

1 Introduction

A directed polygonal path ξ in $\mathbb{R}^d$ is *monotone* if there exists a nonzero vector $\mathbf{u} \in \mathbb{R}^d$ that has a positive inner product with every directed edge of ξ. The study of combinatorial properties of monotone paths is motivated by the classical simplex algorithm in linear programming, which finds an optimal solution by tracing a monotone path in the 1-skeleton of a d-dimensional polytope of feasible solutions. It remains an elusive open problem whether there is a pivoting rule for the simplex method that produces a monotone path whose length is polynomial in d and n [1].

Let S be a set of n points in the plane. A *geometric graph* G is a graph drawn in the plane so that the vertex set consists of the points in S and the edges are drawn as straight line segments between the corresponding points in S. A *plane geometric graph* is one in which edges intersect only at common endpoints. In this paper, we are interested in the maximum number of monotone paths over all plane geometric graphs with n vertices; it is easy to see that triangulations maximize the number of such paths (since adding edges can only increase the number of monotone paths).

Our results. We first show that the number of monotone paths (over all directions) in a triangulation of n points in the plane is $O(1.8193^n)$, using a fingerprinting technique in which incidence patterns of 8 vertices are analyzed. We then give a sharper bound of $O(1.7864^n)$ using the same strategy, by enumerating fingerprints of 11 vertices using a computer program.

Theorem 1. *The number of monotone paths in a geometric triangulation on n vertices in the plane is $O(1.7864^n)$.*

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It is often challenging to determine the number of configurations (i.e., count) faster than listing all such configurations (i.e., enumerate). In Section 6 we show that monotone paths can be counted in polynomial time in plane graphs.

Theorem 2. *Given a plane geometric graph G with n vertices, the number of monotone paths in G can be computed in $O(n^2)$ time. The monotone paths can be enumerated in an additional $O(1)$-time per edge, i.e., in $O(n^2 + K)$ time, where K is the sum of the lengths of all monotone paths.*

Related previous work. We derive a new upper bound on the maximum number of monotone paths in geometric triangulations of n points in the plane. Analogous problems have been studied for cycles, spanning cycles, spanning trees, and matchings [4] in n-vertex edge-maximal planar graphs, which are defined in purely graph theoretic terms. In contrast, the monotonicity of a path depends on the embedding of the point set in the plane, i.e., it is a *geometric* property. The number of geometric configurations contained (as a subgraph) in a triangulation of n points have been considered only recently. The maximum number of *convex polygons* is known to be between $\Omega(1.5028^n)$ and $O(1.5029^n)$ [9, 16]. For the number of *monotone paths*, Dumitrescu et al. [5] gave an upper bound of $O(1.8393^n)$; we briefly review their proof in Section 2. A lower bound of $\Omega(1.7003^n)$ is established in the same paper. It can be deduced from the following construction illustrated in Fig. 1. Let $n = 2^\ell + 2$ for an integer $\ell \in \mathbb{N}$; the plane graph G has n vertices $V = \{v_1, \ldots, v_n\}$, it contains the Hamiltonian path $\xi_0 = (v_1, \ldots, v_n)$, and it has edge (v_i, v_{i+2^k}), for $1 \le i \le n - 2^k$, iff $i - 1$ or $i - 2$ is a multiple of 2^k.

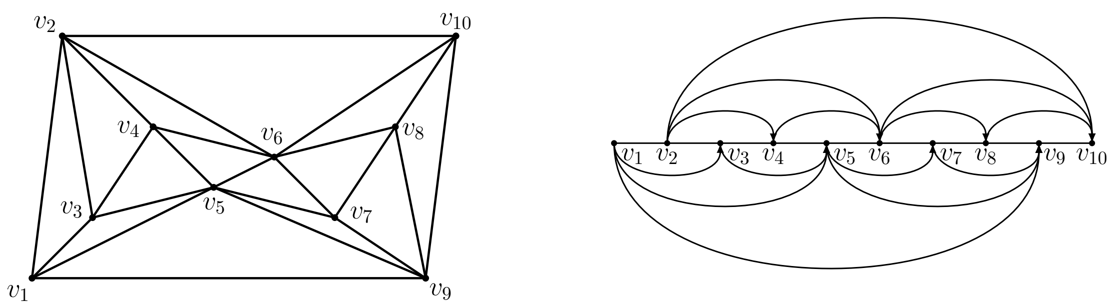

Figure 1: Left: a graph on $n = 2^\ell + 2$ vertices (here $\ell = 3$) that contains a Hamiltonian path $\xi_0 = (v_1, \ldots, v_n)$. Right: an isomorphic plane monotone graph where corresponding vertices are in the same order by x-coordinate; and edges above (resp., below) ξ_0 remain above (resp., below) ξ_0. For n sufficiently large, a graph in this family contains $\Omega(1.7003^n)$ x-monotone paths.

Every n-vertex triangulation contains $\Omega(n^2)$ monotone paths, since there is a monotone path between any two vertices (by a straightforward adaptation of [6, Lemma 1] from convex subdivisions to triangulations). The *minimum* number of monotone paths in an n-vertex triangulation lies between $\Omega(n^2)$ and $O(n^{3.17})$ [5].

The number of several common crossing-free structures (such as matchings, spanning trees, spanning cycles, triangulations) on a set of n points in the plane is known to be exponential [2, 7, 10, 19, 22, 23, 24, 25]; see also [8, 26]. Early upper bounds in this area were obtained by multiplying an upper bound on the maximum number of triangulations on n points with an upper bound on the maximum number of desired configurations in an n-vertex triangulation; valid upper bounds result since every plane geometric graph can be augmented into a triangulation.

For a polytope $P \subset \mathbb{R}^d$, let $G(P)$ denote its *1-skeleton*, which is the graph consisting of the vertices and edges of P. The efficiency of the simplex algorithm and its variants hinges on extremal bounds on the *length* of a monotone paths in the 1-skeleton of a polytope. For example, the *monotone Hirsch conjecture* [29] states that for every $\mathbf{u} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, the 1-skeleton of every d-dimensional polytope with n facets contains a $\mathbf{u}$-monotone path with at most $n-d$ edges from any vertex to a $\mathbf{u}$-maximal vertex (i.e., a vertex whose orthogonal projection onto $\mathbf{u}$ is maximal). Klee [15] verified the conjecture for 3-dimensional polytopes, but counterexamples have been found in dimensions $d \geq 4$ [27] (see also [20]). Kalai [13, 14] gave a subexponential upper bound for the length of a *shortest* monotone path between any two vertices (better bounds are known for the diameter of the 1-skeleta of polyhedra [28], but the shortest path between two vertices need *not* be monotone). However, even in $\mathbb{R}^3$, no deterministic pivot rule is known to find a monotone path of length $n-3$ [12], and the expected length of a path found by randomized pivot rules requires averaging over all $\mathbf{u}$-monotone paths [11, 17]. See also [21] for a summary of results of the `polymath 3` project on the *polynomial Hirsch conjecture*.

2 Preliminaries

A polygonal path $\xi = (v_1, v_2, \ldots, v_t)$ in $\mathbb{R}^d$ is *monotone in direction* $\mathbf{u} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ if every directed edge of ξ has a positive inner product with $\mathbf{u}$, that is, $\langle \overrightarrow{v_i v_{i+1}}, \mathbf{u} \rangle > 0$ for $i = 1, \ldots, t-1$; here $\mathbf{0}$ is the origin. A path $\xi = (v_1, v_2, \ldots, v_t)$ is *monotone* if it is monotone in some direction $\mathbf{u} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$. A path ξ in the plane is *x-monotone*, if it is monotone with respect to the positive direction of the x-axis, i.e., monotone in direction $\mathbf{u} = (1, 0)$.

Let S be a set of n points in the plane. A (geometric) *triangulation* of S is a plane geometric graph with vertex set S such that the bounded faces are triangles that jointly tile of the convex hull of S. Since a triangulation has at most $3n-6$ edges for $n \geq 3$, and the $\mathbf{u}$-monotonicity of an edge (a, b) depends on the sign of $\langle \overrightarrow{ab}, \mathbf{u} \rangle$, it is enough to consider monotone paths in at most $2(3n-6) = 6n-12$ directions (one direction between any two consecutive unit normal vectors of the edges). In the remainder of the paper, we obtain an upper bound on the number of monotone paths in a fixed direction, which we may assume to be the positive direction of the x-axis.

Let $G = (S, E)$ be a plane geometric graph with n vertices. An x-monotone path ξ in G is *maximal* if ξ is not a proper subpath (consisting of consecutive vertices) of some x-monotone path in G. Every x-monotone path in G contains at most n vertices, hence it contains at most $\binom{n}{2}$ x-monotone subpaths. Conversely, every x-monotone path can be extended to a maximal x-monotone path. Let λ_n denote the maximum number of monotone paths in an n-vertex triangulation (summed over all directions $\mathbf{u}$). Let μ_n denote the maximum number (over all directions $\mathbf{u}$) of maximal $\mathbf{u}$-monotone paths in an n-vertex triangulation. As such, we have

$$\lambda_n \leq (6n-12)\binom{n}{2} \cdot \mu_n = O(n^3 \mu_n).$$

We prove an upper bound for a broader class of graphs, *plane monotone graphs*, in which every edge is an x-monotone Jordan arc. Consider a plane monotone graph G on n vertices with a maximum number of x-monotone paths. We may assume that the vertices have distinct x-coordinates; otherwise we can perturb the vertices without decreasing the number of x-monotone paths. Since inserting new edges can only increase the number of x-monotone paths, we may also assume that G is fully triangulated [18, Lemma 3.1], i.e., it is an edge-maximal planar graph. Conversely, every plane monotone graph is isomorphic to a plane geometric graph in which the x-coordinates of the corresponding vertices are the same [18, Theorem 2]. Consequently, the number of maximal x-monotone paths in G equals μ_n.

Denote the vertex set of G by $W = \{w_1, w_2, \ldots, w_n\}$, ordered by increasing x-coordinates; and direct each edge $w_i w_j \in E(G)$ from w_i to w_j if $i < j$; we thereby obtain a directed graph

G. By [5, Lemma 3], all edges $w_i w_{i+1}$ must be present, i.e., G contains a Hamiltonian path $\xi_0 = (w_1, w_2, \ldots, w_n)$. If $T(i)$ denotes the number of maximal (w.r.t. inclusion) x-monotone paths in G starting at vertex w_{n-i+1}, it was shown in the same paper that $T(i)$ satisfies the recurrence $T(i) \leq T(i-1) + T(i-2) + T(i-3)$ for $i \geq 4$, with initial values $T(1) = T(2) = 1$ and $T(3) = 2$ (one-vertex paths are also counted). This recurrence solves to $T(n) = O(\alpha^n)$, where $\alpha = 1.8392\ldots$ is the unique real root of the cubic equation $x^3 - x^2 - x - 1 = 0$. Consequently, any n-vertex geometric triangulation admits at most $O(n^3\, T(n)) = O(1.8393^n)$ monotone paths. Theorem 1 improves this bound to $O(1.7864^n)$.

Fingerprinting technique. An x-monotone path can be represented uniquely by the subset of visited vertices. This unique representation gives the trivial upper bound of 2^n for the number of x-monotone paths. For a set of k vertices $V \subseteq W$, an *incidence pattern* of V (*pattern*, for short) is a subset of V that appears in a monotone path ξ (i.e., the intersection between V and a monotone path ξ). Denote by $I(V)$ the set of all incidence patterns of V; see Fig. 2. For instance, $v_1 v_3 \in I(V)$ implies that there exists a monotone path ξ in G that is incident to v_1 and v_3 in V, but no other vertices in V. The incidence pattern $\emptyset \in I(V)$ denotes an empty intersection between ξ and V, i.e., a monotone path that has no vertices in V.

We now describe a *divide & conquer* application of the fingerprinting technique we use in our proof. For $k \in \mathbb{N}$, let $p_k = \max_{|V|=k} |I(V)|$ denote the maximum number of incidence patterns for a set V of k consecutive vertices in a plane monotone triangulation. We trivially have $p_k \leq 2^k$, and it immediately follows from the definition that $p_k \leq p_i p_j$ for all $i, j \geq 1$ with $i + j = k$; in particular, we have $p_{2k} \leq p_k^2$. Assuming that n is a multiple of k, the product rule yields $\mu_n \leq p_k^{n/k}$. For arbitrary n and constant k, we obtain $\mu_n \leq p_k^{\lfloor n/k \rfloor} 2^{n-k\lfloor n/k \rfloor} \leq p_k^{\lfloor n/k \rfloor} 2^k = O\left(p_k^{n/k}\right)$. Table 1 summarizes the upper bounds obtained by this approach.

k	p_k	$\mu_n = O\left(p_k^{n/k}\right)$	$\lambda_n = O(n^3 \mu_n)$
2	4	2^n	$O(n^3\, 2^n)$
3	7	$O(7^{n/3})$	$O(n^3\, 7^{n/3}) = O(1.913^n)$
4	13	$O(13^{n/4})$	$O(n^3\, 13^{n/4}) = O(1.8989^n)$
5	23	$O(23^{n/5})$	$O(n^3\, 23^{n/5}) = O(1.8722^n)$
6	41	$O(41^{n/6})$	$O(n^3\, 41^{n/6}) = O(1.8570^n)$
7	70	$O(70^{n/7})$	$O(n^3\, 70^{n/7}) = O(1.8348^n)$
8	120	$O(120^{n/8})$	$O(n^3\, 120^{n/8}) = O(1.8193^n)$
9	201	$O(201^{n/9})$	$O(n^3\, 201^{n/9}) = O(1.8027^n)$
10	346	$O(346^{n/10})$	$O(n^3\, 346^{n/10}) = O(1.7944^n)$
11	591	$O(591^{n/11})$	$O(n^3\, 591^{n/11}) = O(1.7864^n)$

Table 1: Upper bounds obtained via the fingerprinting technique for $k \leq 11$.

It is clear that $p_1 = 2$ and $p_2 = 4$, and it is not difficult to see that $p_3 = 7$ (note that $p_3 < p_1 p_2$). We prove $p_4 = 13$ (and so $p_4 < p_2^2 = 16$) by analytic methods (Section 3); this yields the upper bounds $\mu_n = O(13^{n/4})$ and consequently $\lambda_n = O(n^3 13^{n/4}) = O(1.8989^n)$. A careful analysis of the edges between two consecutive groups of 4 vertices shows that $p_8 = 120$, and so p_8 is significantly smaller than $p_4^2 = 13^2 = 169$ (Lemma 10), hence $\mu_n = O(120^{n/8})$ and $\lambda_n = O(n^3 120^{n/8}) = O(1.8193^n)$. Computer search shows that $p_{11} = 591$, and so $\mu_n = O(591^{n/11})$ and $\lambda_n = O(n^3 591^{n/11}) = O(1.7864^n)$ (Section 5).

The analysis of p_k, for $k \geq 12$, using the same technique is expected to yield further improvements. Handling incidence patterns on 12 or 13 vertices is still realistic (although time consuming), but working with larger groups is currently prohibitive, both by analytic methods and with computer search. Significant improvement over our results may require new ideas.

Definitions and notations for a single group. Let G be a directed plane monotone triangulation that contains a Hamiltonian path $\xi_0 = (w_1, w_2, \ldots, w_n)$. Denote by G^- (resp., G^+) the path ξ_0 together with all edges below (resp., above) ξ_0. Let $V = \{v_1, \ldots, v_k\}$ be a set of k consecutive vertices of ξ_0. For the purpose of identifying the edges relevant for the incidence patterns of V, the edges between a vertex $v_i \in V$ and any vertex preceding V (resp., succeeding V) are equivalent since they correspond to the same incidence pattern. We therefore apply a graph homomorphism φ on G^- and G^+, respectively, that maps all vertices preceding V to a new node v_0, and all vertices succeeding V to a new node v_{k+1}. The path ξ_0 is mapped to a new path $(v_0, v_1, \ldots, v_k, v_{k+1})$. Denote the edges in $\varphi(G^- \setminus \xi_0)$ and $\varphi(G^+ \setminus \xi_0)$, respectively, by $E^-(V)$ and $E^+(V)$; they are referred to as the *upper side* and the *lower side*; and let $E(V) = E^-(V) \cup E^+(V)$. The incidence pattern of the vertex set V is determined by the triple $(V, E^-(V), E^+(V))$. We call this triple the *group* induced by V, or simply the *group* V.

Figure 2: Left: a group U with incidence patterns $I(U) = \{\emptyset,\ u_1u_2,\ u_1u_2u_3,\ u_1u_2u_3u_4,\ u_1u_2u_4,\ u_2, u_2u_3,$ $u_2u_3u_4,\ u_2u_4,\ u_3, u_3u_4\}$. Right: a group V with $I(V) = \{\emptyset,\ v_1v_2,\ v_1v_2v_3,\ v_1v_2v_3v_4,\ v_1v_2v_4,\ v_1v_3,\ v_1v_3v_4,\ v_2,$ $v_2v_3,\ v_2v_3v_4,\ v_2v_4,\ v_3,\ v_3v_4\}$.

The edges $v_iv_j \in E(V)$, $1 \leq i < j \leq k$, are called *inner edges*. The edges v_0v_i, $1 \leq i \leq k$, are called *incoming edges* of $v_i \in V$; and the edges v_iv_{k+1}, $1 \leq i \leq k$, are *outgoing edges* of $v_i \in V$ (note that v_0 and v_{k+1} are not in V). An incoming edge v_0v_i for $1 < i \leq k$ (resp., and outgoing edge v_iv_{k+1} for $1 \leq i < k$) may be present in both $E^-(V)$ and $E^+(V)$. Denote by $\texttt{In}(v)$ and $\texttt{Out}(v)$, respectively, the number of incoming and outgoing edges of a vertex $v \in V$; and note that $\texttt{In}(v)$ and $\texttt{Out}(v)$ can be 0, 1 or 2.

For $1 \leq i \leq k$, let V_{*i} denote the set of incidence patterns in the group V ending at i. For example in Fig. 2 (right), $V_{*3} = \{v_1v_2v_3, v_1v_3, v_2v_3, v_3\}$. By definition we have $|V_{*i}| \leq 2^{i-1}$. Similarly V_{i*} denotes the set of incidence patterns in the group V starting at i. In Fig. 2 (left), $U_{2*} = \{u_2, u_2u_3, u_2u_3u_4, u_2u_4\}$. Observe that $|V_{i*}| \leq 2^{k-i}$. Note that

$$|I(V)| = 1 + \sum_{i=1}^{k} |V_{*i}| \qquad \text{and} \qquad |I(V)| = 1 + \sum_{i=1}^{k} |V_{i*}|. \tag{1}$$

Reflecting all components of a triple $(V, E^-(V), E^+(V))$ with respect to the x-axis generates a new group denoted by $(V, E^-(V), E^+(V))^R$, or V^R for a shorthand notation. By definition, both V and V^R have the same set of incidence patterns.

Remark. Our counting arguments pertain to maximal x-monotone paths. Suppose that a maximal x-monotone path ξ has an incidence pattern in V_{*i}, for some $1 \leq i < k$. By the maximality of ξ, ξ must leave the group after v_i, and so v_i must be incident to an outgoing edge. Similarly, the existence of a pattern in V_{i*} for $1 < i \leq k$, implies that v_i is incident to an incoming edge.

3 Groups of 4 vertices

In this section we analyze the incidence patterns of groups with 4 vertices. We prove that $p_4 = 13$ and find the only two groups with 4 vertices that have 13 patterns (Lemma 5). We also prove important properties of groups that have exactly 11 or 12 patterns, respectively (Lemmata 2, 3 and 4).

Lemma 1. *Let V be a group of 4 vertices with at least 10 incidence patterns. Then there is*

(i) *an outgoing edge from v_2 or v_3; and*

(ii) *an incoming edge into v_2 or v_3.*

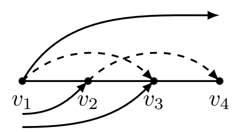

Figure 3: v_1 cannot be the last vertex with an outgoing edge from a group $V = \{v_1, v_2, v_3, v_4\}$ with at least 10 incidence patterns.

Proof. (i) There is at least one outgoing edge from $\{v_1, v_2, v_3\}$, since otherwise $V_{*1} = V_{*2} = V_{*3} = \emptyset$ implying $|I(V)| = |V_{*4}| + 1 \leq 9$. Assume there is no outgoing edge from v_2 and v_3; then $V_{*1} = \{v_1\}$ and $V_{*2} = V_{*3} = \emptyset$. From (1), we have $|V_{*4}| = 8$ and this implies $\{v_1v_3v_4, v_2v_4, v_3v_4\} \subset V_{*4}$. The patterns $v_1v_3v_4$ and v_2v_4, respectively, imply that $v_1v_3, v_2v_4 \in E(V)$. The patterns v_2v_4 and v_3v_4, respectively, imply there are incoming edges into v_2 and v_3. Refer to Fig. 3. Without loss of generality, an outgoing edge from v_1 is in $E^+(V)$. By planarity, all incoming edges into v_2 or v_3 have to be in $E^-(V)$. Then v_1v_3 and v_2v_4 both have to be in $E^+(V)$, which by planarity is impossible.

(ii) By symmetry in a vertical axis, there is an incoming edge into v_2 or v_3. □

Lemma 2. *Let V be a group of 4 vertices with at least 11 incidence patterns. Then there is*

(i) *an incoming edge into v_2; and*

(ii) *an outgoing edge from v_3.*

Proof. (i) Assume $\texttt{In}(v_2) = 0$. Then $|V_{2*}| = 0$. By Lemma 1 (ii), we have $\texttt{In}(v_3) > 0$. By definition $|V_{3*}| \leq 2$. We distinguish two cases.

Case 1: $\texttt{In}(v_4) = 0$. In this case, $|V_{4*}| = 0$. Refer to Fig. 4 (left). By planarity, the edge v_1v_4 and an outgoing edge from v_2 cannot coexist with an incoming edge into v_3. So either v_1v_4 or v_1v_2 is not in V_{1*}, which implies $|V_{1*}| < 8$. Therefore, (1) yields $|I(V)| = |V_{1*}| + |V_{3*}| + 1 < 8 + 2 + 1 = 11$, which is a contradiction.

Case 2: $\texttt{In}(v_4) > 0$. In this case, $|V_{4*}| = 1$. If the incoming edges into v_3 and v_4 are on opposite sides (see Fig. 4 (center)), then by planarity there are outgoing edges from neither v_1 nor v_2, which implies that the patterns v_1 and v_1v_2 are not in V_{1*}, and so $|V_{1*}| \leq 8 - 2 = 6$. If the incoming edges into v_3 and v_4 are on the same side (see Fig. 4 (right)), then by planarity either the edges v_1v_4 and v_2v_4 or an outgoing edge from v_3 cannot exist, which implies that either v_1v_4 and $v_1v_2v_4$ are not in V_{1*} or v_1v_3 and $v_1v_2v_3$ are not in V_{1*}. In both cases, $|V_{1*}| \leq 8 - 2 = 6$.

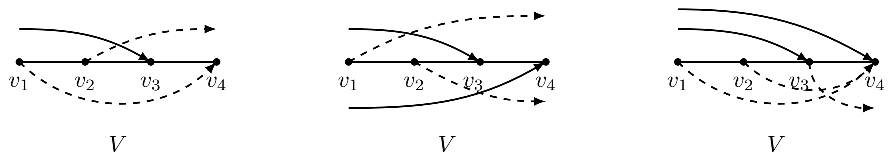

Figure 4: Left: an incoming edge arrives into v_3, but not into v_4. Center and right: incoming edges arrive into both v_3 and v_4; either on the same or on opposite sides of ξ_0.

Therefore, irrespective of the relative position of the incoming edges into v_3 and v_4 (on the same side or on opposite sides), (1) yields $|I(V)| = |V_{1*}| + |V_{3*}| + |V_{4*}| + 1 \leq 6 + 2 + 1 + 1 = 10$, which is a contradiction.

(ii) By symmetry in a vertical axis, $\texttt{Out}(v_3) > 0$. □

Lemma 3. *Let V be a group of 4 vertices with exactly 11 incidence patterns. Then the following hold.*

(i) *If $\texttt{In}(v_3) = 0$, then all the incoming edges into v_2 are on the same side of ξ_0, $|V_{1*}| \geq 5$, and $|V_{2*}| \geq 3$.*

(ii) *If $\texttt{In}(v_3) > 0$, then all the incoming edges into v_3 are on the same side of ξ_0, $|V_{1*}| \geq 4$, $|V_{2*}| \geq 2$, and $|V_{3*}| = 2$.*

Proof. By Lemma 2, $\texttt{In}(v_2) \neq 0$ and $\texttt{Out}(v_3) \neq 0$. Therefore $\{v_2v_3, v_2v_3v_4\} \subseteq V_{2*}$, implying $|V_{2*}| \geq 2$. By definition $|V_{4*}| \leq 1$.

(i) Assume that $\texttt{In}(v_3) = 0$. Then we have $|V_{3*}| = 0$. By (1), we obtain $|V_{1*}| + |V_{2*}| \geq 9$. By definition $|V_{2*}| \leq 4$, implying $|V_{1*}| \geq 5$. All incoming edges into v_2 are on the same side, otherwise the patterns $\{v_1, v_1v_3, v_1v_3v_4, v_1v_4\}$ cannot exist, which would imply $|V_{1*}| < 5$. If $|V_{2*}| < 3$, then v_2 and v_2v_4 are not in V_{2*} implying that v_1v_2 and $v_1v_2v_4$ are not in V_{1*}; hence $|V_{1*}| \leq 6$ and thus $|V_{1*}| + |V_{2*}| < 9$, which is a contradiction. We conclude that $|V_{2*}| \geq 3$.

(ii) Assume that $\texttt{In}(v_3) > 0$. Then we have $\{v_3, v_3v_4\} \subseteq V_{3*}$, hence $|V_{3*}| = 2$. By (1), we obtain $|V_{1*}| + |V_{2*}| \geq 7$. If $|V_{1*}| < 4$, then $|V_{2*}| \geq 4$ and so $\{v_2, v_2v_3, v_2v_4, v_2v_3v_4\} \subseteq V_{2*}$. This implies $\{v_1v_2, v_1v_2v_3, v_1v_2v_4, v_1v_2v_3v_4\} \subseteq V_{1*}$, hence $|V_{1*}| \geq 4$, which is a contradiction. We conclude that $|V_{1*}| \geq 4$. All incoming edges into v_3 are on the same side, otherwise the patterns $\{v_1, v_1v_2, v_1v_2v_4, v_1v_4, v_2, v_2v_4\}$ cannot exist, and thus $|I(V)| \leq 10$, which is a contradiction. □

Lemma 4. *Let V be a group of 4 vertices with exactly 12 incidence patterns. Then the following hold.*

(i) *For $i = 1, 2, 3$, all outgoing edges from v_i, if any, are on the same side of ξ_0.*

(ii) *If V has outgoing edges from exactly one vertex, then this vertex is v_3 and we have $|V_{*3}| = 4$ and $|V_{*4}| = 7$. Otherwise there are outgoing edges from v_2 and v_3, and we have $|V_{*2}| = 2$, $|V_{*3}| \geq 3$ and $|V_{*4}| \geq 5$.*

(iii) *For $i = 2, 3, 4$, all incoming edges into v_i, if any, are on one side of ξ_0.*

(iv) *If V has incoming edges into exactly one vertex, then this vertex is v_2 and we have $|V_{2*}| = 4$ and $|V_{1*}| = 7$. Otherwise there are incoming edges into v_3 and v_2, and we have $|V_{3*}| = 2$, $|V_{2*}| \geq 3$ and $|V_{1*}| \geq 5$.*

Proof. (i) By Lemma 2 (i), there is an incoming edge into v_2. So by planarity, all outgoing edges from v_1, if any, are on one side of ξ_0.

If there are outgoing edges from v_2 on both sides, then by planarity the edges v_1v_3, v_1v_4 and any incoming edge into v_3 cannot exist, hence the five patterns $\{v_1v_3, v_1v_3v_4, v_1v_4, v_3, v_3v_4\}$ are not in $I(V)$ and thus $|I(V)| \le 16 - 5 = 11$, which is a contradiction.

If there are outgoing edges from v_3 on both sides (see Fig. 5 (a)), then by planarity the edges v_1v_4, v_2v_4 and an incoming edge into v_4 cannot exist, hence the four patterns $\{v_1v_2v_4, v_1v_4, v_2v_4, v_4\}$ are not in $I(V)$. Without loss of generality, an incoming edge into v_2 is in $E^+(V)$. Then by planarity, any outgoing edge of v_1 and the edge v_1v_3 (which must be present) are in $E^-(V)$. Also by planarity, either an incoming edge into v_3 or an outgoing edge from v_2 cannot exist. So either the patterns $\{v_3, v_3v_4\}$ or the patterns $\{v_1v_2, v_2\}$ are not in $I(V)$. Hence $|I(V)| \le 16 - (4 + 2) = 10$, which is a contradiction. Consequently, all outgoing edges of v_i are on the same side of ξ_0, for $i = 1, 2, 3$.

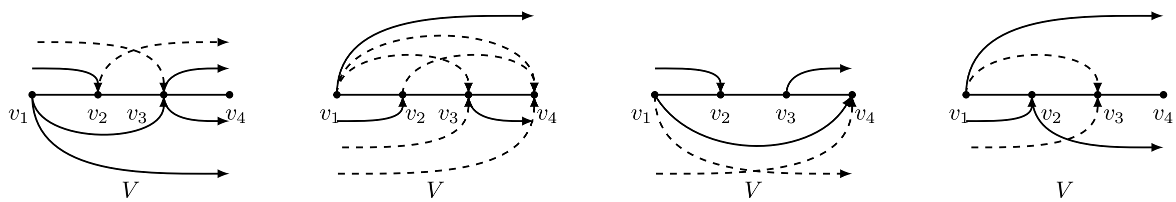

Figure 5: (a) Having outgoing edges from v_3 on both sides is impossible. (b) Existence of outgoing edges only from $\{v_1v_3\}$ is impossible. (c) $|V_{*3}| \ge 3$. (d) $|V_{*4}| \ge 5$.

(ii) If V has outgoing edges from exactly one vertex, then by Lemma 2 (ii), this vertex is v_3. Consequently, $V_{*1} = V_{*2} = \emptyset$. Using (1), $|V_{*3}| + |V_{*4}| = 11$. Therefore $|V_{*4}| \ge 7$, since by definition $|V_{*3}| \le 4$. If $|V_{*4}| = 8$, then we have $\{v_1v_2v_3v_4, v_1v_3v_4, v_2v_3v_4, v_3v_4\} \subset V_{*4}$. Existence of these four patterns along with an outgoing edge from v_3 implies $\{v_1v_2v_3, v_1v_3, v_2v_3, v_3\} \subseteq V_{*3}$ and thus $|V_{*3}| + |V_{*4}| = 4 + 8 = 12$, which is a contradiction. Therefore $|V_{*4}| = 7$ and $|V_{*3}| = 4$.

If V has outgoing edges from more than one vertex, the possible vertex sets with outgoing edges are $\{v_1, v_3\}$, $\{v_2, v_3\}$, and $\{v_1, v_2, v_3\}$. We show that it is impossible that all outgoing edges are from $\{v_1, v_3\}$, which will imply that there are outgoing edges from both v_2 and v_3.

If there are outgoing edges from $\{v_1, v_3\}$ only, we may assume the ones from v_1 are in $E^+(V)$ and then by planarity all incoming edges into v_2 are in $E^-(V)$, see Fig. 5 (b). Then by planarity, either v_1v_3 or v_2v_4 or an incoming edge into v_3 cannot exist implying that $\{v_1v_3, v_1v_3v_4\}$ or $\{v_1v_2v_4, v_2v_4\}$ or $\{v_3, v_3v_4\}$ is not in $I(V)$. By the same token, depending on the side the outgoing edges from v_3 are on, either the edge v_1v_4 or an incoming edge into v_4 cannot exist, implying that either v_1v_4 or v_4 is not in $I(V)$. Since $V_{*2} = \emptyset$, $\{v_1v_2, v_2\}$ are not in $I(V)$. So $|I(V)| \le 16-(2+1+2) = 11$, which is a contradiction. Therefore the existence of outgoing edges only from v_1 and v_3 is impossible.

If there are outgoing edges from (precisely) $\{v_2, v_3\}$ or $\{v_1, v_2, v_3\}$, then we have $\{v_1v_2, v_2\} \subseteq V_{*2}$ and $\{v_1v_2v_3, v_2v_3\} \subseteq V_{*3}$, since $\texttt{In}(v_2) \ne 0$ and $\texttt{Out}(v_3) \ne 0$ by Lemma 2. Therefore $|V_{*2}| = 2$ and $|V_{*3}| \ge 2$. If $|V_{*3}| < 3$, then $v_1v_3, v_3 \notin V_{*3}$, which implies that v_1v_3 and an incoming edge into v_3 are not in $E(V)$. Consequently, $v_1v_3v_4$, $v_3v_4 \notin I(V)$. Observe Fig. 5 (c). By planarity the edge v_1v_4, an incoming edge into v_4 and an outgoing edge from v_1 cannot exist together with an incoming edge into v_2 and an outgoing edge from v_3. So at least one of the patterns $\{v_1, v_1v_4, v_4\}$ is missing implying $|I(V)| \le 16 - (2 + 2 + 1) = 11$, which is a contradiction. So $|V_{*3}| \ge 3$. If $|V_{*4}| < 5$, then (1) yields $|V_{*3}| = 4$, $|V_{*2}| = 2$ and $|V_{*1}| = 1$. We may assume that all outgoing edges from v_1 are in $E^+(V)$; see Fig. 5 (d). By planarity, the incoming edges into v_2 are in $E^-(V)$. Depending on the side the outgoing edges from v_2 are on, either v_1v_3 or an incoming edge into v_3 cannot exist, implying that either v_1v_3 or v_3 is not in V_{*3}, therefore $|V_{*3}| < 4$, creating a contradiction. We conclude that $|V_{*4}| \geq 5$.

(iii) By symmetry, (iii) immediately follows from (i).

(iv) By symmetry, (iv) immediately follows from (ii). $\square$

Lemma 5. *Let V be a group of 4 vertices. Then V has at most 13 incidence patterns. If V has 13 incidence patterns, then V is either A or A^R in Fig. 6. Consequently, $p_4 = 13$.*

Figure 6: $I(A) = I(A^R) = \{\emptyset, 12, 123, 1234, 124, 13, 134, 2, 23, 234, 24, 3, 34\}$. A and A^R are the only groups with 13 incidence patterns.

Proof. Observe that group A in Fig. 6 has 13 patterns. Let V be a group of 4 vertices with at least 13 patterns.

We first **claim** that V has an incoming edge into v_3 and an outgoing edge from v_2. Their existence combined with Lemma 2 implies that $\{v_3v_4, v_3\} \subset I(V)$ and $\{v_1v_2, v_2\} \subset I(V)$, respectively. At least one of these two edges has to be in $E(V)$, otherwise V has at most $16 - (2 + 2) = 12$ patterns. Assume that one of the two, without loss of generality, the outgoing edge from v_2 is not in $E(V)$. Then $\{v_1v_3, v_2v_4\} \subseteq E(V)$, otherwise either patterns $\{v_1v_3, v_1v_3v_4\}$ or $\{v_1v_2v_4, v_2v_4\}$ are not in $I(V)$ and there are at most $16 - (2 + 2) = 12$ patterns. By Lemma 2, there is an incoming edge into v_2 and an outgoing edge from v_3. Without loss of generality, the outgoing edge from v_3 is in $E^-(V)$. So by planarity v_2v_4 is in $E^+(V)$, which implies that v_1v_3 and the incoming edge into v_3 are in $E^-(V)$. By the same token, the incoming edge into v_2 is in $E^+(V)$. So by planarity the edge v_1v_4 and an outgoing edge from v_1 cannot be in $E(V)$. Then the patterns $\{v_1v_4, v_1\}$ are not in $I(V)$, thus V has at most $16 - (2 + 2) = 12$ patterns, which is a contradiction. This completes the proof of the claim.

We may assume without loss of generality (by applying a reflection in the x-axis if necessary) that the incoming edge into v_3 is in $E^-(V)$. By planarity, the outgoing edge from v_2 is in $E^+(V)$. By the same token the edge v_1v_4 cannot be in $E(V)$, which implies that $v_1v_4 \notin I(V)$. So $I(V) \leq 16 - 1 = 15$. By Lemma 2, there is an incoming edge into v_2 and an outgoing edge from v_3. By planarity, if the incoming edge into v_2 is in $E^+(V)$ then the outgoing edge from v_1 cannot be in $E(V)$, therefore the pattern v_1 is not in $I(V)$. But if the incoming edge into v_2 is in $E^-(V)$ then the edge v_1v_3 cannot be in $E(V)$ therefore neither v_1v_3 nor $v_1v_3v_4$ is in $I(V)$. By a similar argument, if the outgoing edge from v_3 is in $E^-(V)$ then the incoming edge into v_4 cannot be in $E(V)$, therefore the pattern v_4 is not in $I(V)$. But if the outgoing edge from v_3 is in $E^+(V)$ then the edge v_2v_4 cannot be in $E(V)$, therefore neither v_2v_4 nor $v_1v_2v_4$ is in $I(V)$. Since $I(V) \geq 13$, the only solution is $v_1 \notin I(V)$ and $v_4 \notin I(V)$. Therefore V induces the group A and has exactly 13 patterns.

If the incoming edge into v_3 is in $E^+(V)$, then V induces A^R (with exactly 13 patterns). $\square$

4 Groups of 8 vertices

In this section, we analyze two consecutive groups, U and V, each with 4 vertices, and show that $p_8 = 120$ (Lemma 10). Let $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, v_3, v_4\}$, and put $UV = U \cup V$

for short. We may assume that $|I(V)| \le |I(U)|$ (by applying a reflection about the vertical axis if necessary), and we have $|I(U)| \le 13$ by Lemma 5. This yields a trivial upper bound $|I(UV)| \le |I(U)| \cdot |I(V)| \le 13^2 = 169$. It is enough to consider cases in which $10 \le |I(V)| \le |I(U)| \le 13$, otherwise the trivial bound is already less than 120.

In all cases where $|I(U)| \cdot |I(V)| > 120$, we improve on the trivial bound by finding edges between U and V that cannot be present in the group UV. If edge $u_i v_j$ is not in $E(UV)$, then any of the $|U_{*i}| \cdot |V_{j*}|$ patterns that contain $u_i v_j$ is excluded. Since every maximal x-monotone path has at most one edge between U and V, distinct edges $u_i v_j$ exclude disjoint sets of patterns, and we can use the sum rule to count the excluded patterns. We continue with a case analysis.

Lemma 6. *Consider a group UV consisting of two consecutive groups of 4 vertices, where $|I(U)| \ge 10$ and $|I(V)| = 10$. Then UV allows at most 120 incidence patterns.*

Proof. If U has at most 12 patterns, then UV has at most $12 \times 10 = 120$ patterns, and the proof is complete. We may thus assume that U has 13 patterns. By Lemma 5, U is either A or A^R. We may assume, by reflecting UV about the horizontal axis if necessary, that U is A. Refer to Fig. 7 (left). Therefore $|U_{*2}| = 2$, $|U_{*3}| = 4$ and $|U_{*4}| = 6$, according to Figure 6. The cross product of the patterns of U and V produce $13 \times 10 = 130$ possible patterns. We show that at least 10 of them are incompatible in each case. It follows that $|I(UV)| \le 130 - 10 = 120$. Let v_i denote the first vertex with an incoming edge in $E(V)$, where $i \ne 1$. By Lemma 1 (ii), $i = 2$ or 3.

Case 1: $(u_4, v_i) \in E(UV)$. We first show that $|V_{1*}| \ge 3$. By definition $|V_{3*}| \le 2$ and $|V_{4*}| \le 1$. By (1), $|V_{1*}| + |V_{2*}| \ge 9 - (2+1) = 6$. If $|V_{2*}| \le 3$, then $|V_{1*}| \ge 3$. Otherwise $|V_{2*}| = 4$ implying $V_{2*} = \{v_2v_3v_4, v_2v_3, v_2v_4, v_2\}$. This implies there are outgoing edges from v_2 and v_3 in $E(V)$. Therefore $\{v_1v_2v_3v_4, v_1v_2v_3, v_1v_2\} \subset V_{1*}$ and $|V_{1*}| \ge 3$.

Case 1.1: $(u_4, v_i) \in E^-(UV)$; see Fig. 7 (right). As $i = 2$ or 3, by planarity $(u_3, v_1) \notin E(UV)$. Hence at least $|U_{*3}|\ |V_{1*}| \ge 4 \times 3 = 12$ combinations are incompatible.

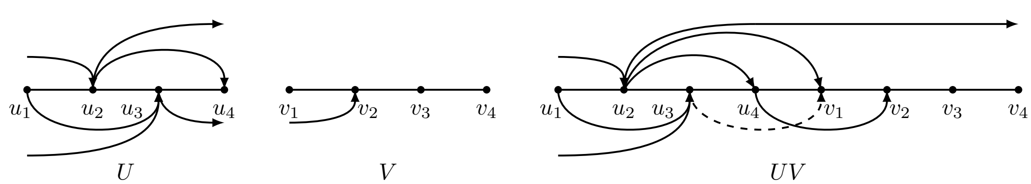

Figure 7: Left: $|I(U)| = 13$ and $|I(V)| = 10$. Right: $(u_4, v_i) \in E^-(UV)$; $i = 2$ here.

Case 1.2: $(u_4, v_i) \in E^+(UV)$; see Fig. 8 (right). Then by planarity $(u_2, v_1) \notin E(UV)$ and $|U_{*2}||V_{1*}| \ge 2 \times 3 = 6$ combinations are incompatible.

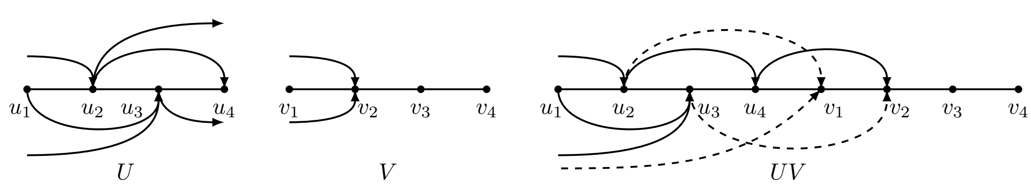

Figure 8: Left: $|I(U)| = 13$ and $|I(V)| = 10$. Right: $(u_4, v_i) \in E^+(UV)$; $i = 2$ here.

An incoming edge into v_i in $E(V)$ implies $|V_{i*}| \ge 1$. If $u_3 v_i \notin E(UV)$, then $|U_{*3}||V_{i*}| \ge 4 \times 1 = 4$ combinations are incompatible. Hence there are at least $6 + 4 = 10$ incompatible patterns. If $u_3 v_i \in E(UV)$, then by planarity an incoming edge into v_1 in $E(UV)$ cannot exist and $|\{\emptyset\}||V_{1*}| \geq 1 \times 3 = 3$ combinations are incompatible. If $u_2v_i \in E(UV)$, then by planarity an outgoing edge from u_4 cannot exist. So $|U_{*4}||\{\emptyset\}| \geq 6 \times 1 = 6$ combinations are incompatible. So there are at least $6+3+6 = 15$ incompatible patterns. If $u_2v_i \notin E(UV)$, then $|U_{*2}||V_{1*}| \geq 2\times 1 = 2$ combinations are incompatible. Hence there are at least $6 + 3 + 2 = 11$ incompatible patterns.

Case 2: $(u_4, v_i) \notin E(UV)$. By showing $|V_{i*}| \geq 2$ for all possible values of i (i.e., 2 and 3), we can conclude that at least $|U_{*4}||V_{i*}| \geq 6 \times 2 = 12$ combinations are incompatible.

If $i = 2$, then $v_2v_3v_4 \in V_{2*}$. By Lemma 1 (i), there is an outgoing edge from v_2 or v_3 in $E(V)$, which implies $v_2 \in V_{2*}$ or $v_2v_3 \in V_{2*}$. Hence $|V_{2*}| \geq 2$.

If $i = 3$ and there is no outgoing edge from v_3 in $E(V)$, then by Lemma 1 (i), there is an outgoing edge from v_2. In that case by planarity, there are only 7 possible incidence patterns $\{\emptyset, v_1v_2v_3v_4, v_1v_2v_4, v_1v_3v_4, v_1v_2, v_3v_4, v_4\}$ in V, which is a contradiction. So if $i = 3$, then there is an outgoing edge from v_3 in $E(V)$, which implies $\{v_3v_4, v_3\} \subset V_{3*}$ therefore $|V_{3*}| \geq 2$. □

Lemma 7. *Consider a group UV consisting of two consecutive groups of 4 vertices, where $|I(U)| \geq 11$ and $|I(V)| = 11$. Then UV allows at most 120 incidence patterns.*

Proof. We distinguish three cases depending on $|I(U)|$.

Case 1: $|I(U)| = 11$. Since $|I(UV)| \leq |I(U)| \cdot |I(V)| = 11 \times 11 = 121$, it suffices to show that at least one of these patterns is incompatible. By Lemma 2, there is an outgoing edge from u_3 in $E(U)$ and an incoming edge into v_2 in $E(V)$. Therefore $u_1u_2u_3 \in U_{*3}$ and $v_2v_3v_4 \in V_{2*}$. Refer to Fig. 9 (left). If $(u_3v_2) \notin E(UV)$, then $u_1u_2u_3v_2v_3v_4$ is not in $I(UV)$. If $(u_3v_2) \in E(UV)$, then by planarity either an outgoing edge from u_4 w.r.t. UV, or an incoming edge into v_1 w.r.t. UV, cannot be in $E(UV)$, implying that either $u_1u_2u_3u_4$ or $v_1v_2v_3v_4$ is not in $I(UV)$.

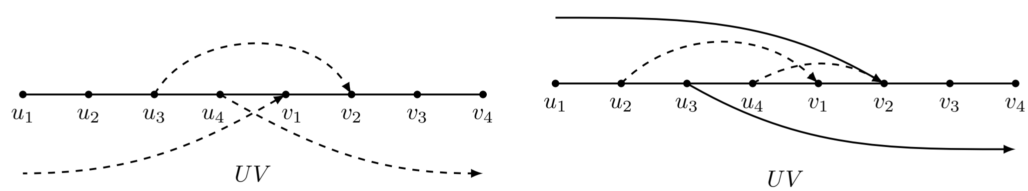

Figure 9: Left: $|I(U)| = |I(V)| = 11$. Right: outgoing edge from u_3 is in $E^-(UV)$ and outgoing edge from u_2 is in $E^+(UV)$.

Case 2: $|I(U)| = 12$. By Lemma 4 (ii), if U has outgoing edges from exactly one vertex, then they are from u_3 and we have $|U_{*3}| = 4$, $|U_{*4}| = 7$, otherwise $|U_{*3}| \geq 3$ and $|U_{*4}| \geq 5$. By Lemma 4 (i), all the outgoing edges from u_3 in $E(U)$ are on one side of U. For simplicity assume those are in $E^-(U)$. Since $|I(UV)| \leq |I(U)| \cdot |I(V)| = 12 \times 11 = 132$, it suffices to show that at least $132 - 120 = 12$ of these patterns are incompatible.

Case 2.1: There is no incoming edge into v_3 in $E(V)$. Then by Lemma 3 (i), all the incoming edges into v_2 in $E(V)$ are on one side of V and we have $|V_{1*}| \geq 5$ and $|V_{2*}| \geq 3$.

Case 2.1.1: The incoming edges into v_2 w.r.t. V are in $E^+(V)$. So by planarity $u_3v_2 \notin E(UV)$ and at least $|U_{*3}||V_{2*}|$ patterns are incompatible. If U has outgoing edges from exactly one vertex, then $|U_{*3}||V_{2*}| \geq 4 \times 3 = 12$ and we are done. Otherwise U has outgoing edges from u_2, where $|U_{*2}| = 2$ and at least $|U_{*3}||V_{2*}| \geq 3 \times 3 = 9$ patterns are incompatible. Also by Lemma 4 (i), all the outgoing edges from u_2 in $E(U)$ are on one side of U. If the outgoing edges from u_2 w.r.t. U are in $E^+(U)$, see Fig. 9 (right), then u_2v_1 and u_4v_2 can only be in $E^+(UV)$; by planarity both edges cannot be in $E(UV)$ and thus at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{2*}|) \geq \min(2\times 5, 5\times 3) = 10$ patterns are incompatible. If the outgoing edges from u_2 w.r.t. U are in $E^-(U)$, then by planarity $u_2v_2 \notin E(UV)$ and thus at least $|U_{*2}||V_{2*}| \geq 2 \times 3 = 6$ patterns are incompatible. Therefore irrespective of the relative position of the outgoing edge from u_2 in $E(U)$, at least $9 + \min(10, 6) = 15$ patterns are incompatible and we are done.

Case 2.1.2: The incoming edges into v_2 w.r.t. V are in $E^-(V)$. Therefore u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(3 \times 5, 5 \times 3) = 15$ patterns are incompatible.

Case 2.2: There is an incoming edge into v_3 in $E(V)$. By Lemma 3 (ii), all the incoming edges into v_3 in $E(V)$ are on one side of V, $|V_{1*}| \geq 4$, $|V_{2*}| \geq 2$ and $|V_{3*}| = 2$.

Case 2.2.1: The incoming edges into v_2 in $E(V)$ are on both sides of V.

If the incoming edges into v_3 w.r.t. V are in $E^+(V)$, see Fig. 10 (left), then by planarity $u_3v_3 \notin E(UV)$. So at least $|U_{*3}||V_{3*}| \geq 3 \times 2 = 6$ patterns are incompatible. By planarity an outgoing edge from u_4 w.r.t. UV, an incoming edge into v_3 w.r.t. UV and u_3v_2 cannot coexist in $E(UV)$. Therefore at least $\min(|\{\emptyset\}||V_{3*}|, |U_{*4}||\{\emptyset\}|, |U_{*3}||V_{2*}|) \geq \min(1 \times 2, 5 \times 1, 3 \times 2) = 2$ patterns are incompatible. By the same argument, the edges u_3v_2, u_4v_2 and an incoming edge into v_1 w.r.t. UV cannot be in $E(UV)$ together. Hence at least $\min(|U_{*3}||V_{2*}|, |U_{*4}||V_{2*}|, |\{\emptyset\}||V_{1*}|) = \min(3 \times 2, 5 \times 2, 1 \times 4) = 4$ patterns are incompatible. Therefore at least $6 + 2 + 4 = 12$ patterns are incompatible.

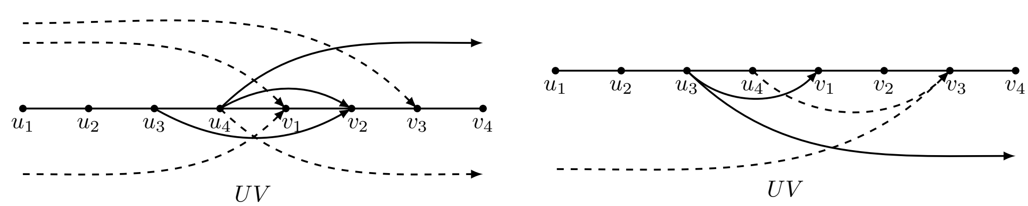

Figure 10: Left: incoming edges into v_2 are in both $E^+(V)$ and $E^-(V)$ and incoming edge into v_3 is in $E^+(V)$. Right: incoming edges into v_2 are in both $E^+(V)$ and $E^-(V)$ and incoming edge into v_3 is in $E^-(V)$.

If incoming edges into v_3 w.r.t. V are in $E^-(V)$, see Fig. 10 (right), then either an outgoing edge from u_3 w.r.t. UV or an incoming edge into v_3 w.r.t. UV cannot be in $E(UV)$. So at least $\min(|U_{*3}||\{\emptyset\}|, |\{\emptyset\}||V_{3*}|) \geq \min(3 \times 1, 1 \times 2) = 2$ patterns are incompatible. Also u_3v_1 and u_4v_3 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{3*}|) \geq \min(3 \times 4, 5 \times 2) = 10$ patterns are incompatible. Therefore at least $2 + 10 = 12$ patterns are incompatible.

Case 2.2.2: All the incoming edges into v_2 in $E(V)$ are on one side of V and the incoming edges into v_2 and v_3 in $E(V)$ are on same side of V.

If the incoming edges into v_2 and v_3 w.r.t. V are in $E^+(V)$, see Fig. 11 (left), then by planarity u_3v_2 and u_3v_3 are not in $E(UV)$. So at least $|U_{*3}||V_{2*}| + |U_{*3}||V_{3*}| \geq 3 \times 2 + 3 \times 2 = 12$ patterns are incompatible.

If the incoming edges into v_2 and v_3 w.r.t. V are in $E^-(V)$, see Fig. 11 (right), then u_3v_1 and both u_4v_2 and u_4v_3 can only be in $E^-(UV)$. By planarity either u_3v_1 or both u_4v_2 and u_4v_3 are not in $E(UV)$. Consequently, at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}| + |U_{*4}||V_{3*}|) = \min(3 \times 4, 5 \times 2 + 5 \times 2) = 12$ patterns are incompatible.

Case 2.2.3: All the incoming edges into v_2 in $E(V)$ are on one side of V and all the incoming edges into v_3 in $E(V)$ are on the opposite side of V. Let the incoming edges w.r.t. V in $E^+(V)$ are into v_i and the incoming edges w.r.t. V in $E^-(V)$ are into v_j. So either $i = 2$, $j = 3$ or

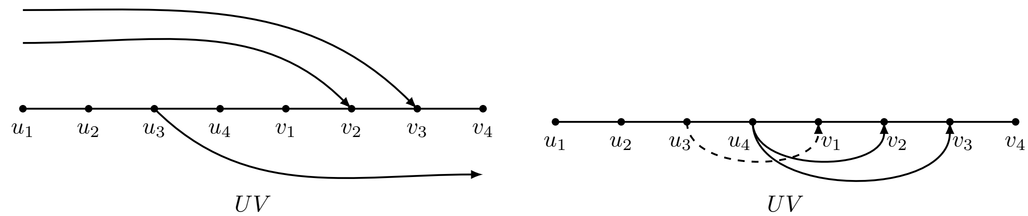

Figure 11: Left: both incoming edges into v_2 and v_3 are in $E^+(V)$. Right: both incoming edges into v_2 and v_3 are in $E^-(V)$.

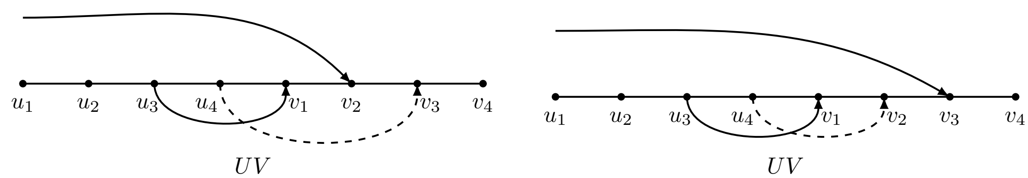

Figure 12: Left: incoming edge into v_2 is $E^+(V)$ and incoming edge into v_3 is in $E^-(V)$. Right: incoming edge into v_2 is in $E^-(V)$ and incoming edge into v_3 is in $E^+(V)$.

$i = 3$, $j = 2$, see Fig. 12. Therefore $|V_{i*}|$, $|V_{j*}|$ are at least $\min(|V_{2*}|, |V_{3*}|) = 2$. By planarity $u_3v_i \notin E(UV)$. So at least $|U_{*3}||V_{i*}| \geq 3 \times 2 = 6$ patterns are incompatible. Also u_3v_1 and u_4v_j can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{j*}|) = \min(3 \times 4, 5 \times 2) = 10$ patterns are incompatible. Therefore at least $6 + 10 = 16$ patterns are incompatible.

Case 3: $|I(U)| = 13$. By Lemma 5, U is either A or A^R. We may assume, by reflecting UV about the horizontal axis if necessary, that U is A. Therefore $|U_{*2}| = 2$, $|U_{*3}| = 4$ and $|U_{*4}| = 6$, see Figure 6. Since $|I(UV)| \leq |I(U)| \cdot |I(V)| = 13 \times 11 = 143$, it suffices to show that at least $143 - 120 = 23$ of these patterns are incompatible.

Case 3.1: There is no incoming edge into v_3 in $E(V)$. Then by Lemma 3 (i), all the incoming edges into v_2 in $E(V)$ are on one side of V, $|V_{1*}| \geq 5$ and $|V_{2*}| \geq 3$.

If the incoming edges into v_2 w.r.t. V are in $E^-(V)$, see Fig. 13 (left), by planarity $u_2v_2 \notin E(UV)$. So at least $|U_{*2}||V_{2*}| \geq 2 \times 3 = 6$ patterns are incompatible. Also u_4v_2 and u_3v_1 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{*4}||V_{2*}|, |U_{*3}||V_{1*}|) = \min(6 \times 3, 4 \times 5) = 18$ patterns are incompatible. Therefore at least $6 + 18 = 24$ patterns are incompatible.

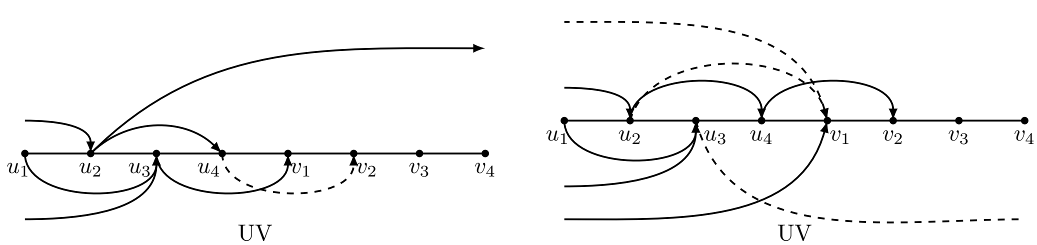

Figure 13: Left: incoming edge into v_2 is in $E^-(V)$. Right: incoming edge into v_2 is in $E^+(V)$.

Similarly if the incoming edges into v_2 w.r.t. V are in $E^+(V)$, cf. Fig. 13 (right), by planarity $u_3v_2 \notin E(UV)$. So at least $|U_{*3}||V_{2*}| \geq 4 \times 3 = 12$ patterns are incompatible. Also u_4v_2 and u_2v_1

can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. If $u_4v_2 \notin E(UV)$, then at least $|U_{*4}||V_{2*}| \geq 6 \times 3 = 18$ patterns are incompatible. Otherwise $u_2v_1 \notin E(UV)$ and either an incoming edge into v_1 w.r.t. UV or an outgoing edge from u_3 w.r.t. UV cannot be in $E(UV)$. Hence at least

$$|U_{*2}||V_{1*}| + \min(|\{\emptyset\}||V_{1*}|, |U_{*3}||\{\emptyset\}|) \geq 2 \times 5 + \min(1 \times 5, 4 \times 1) = 14$$

patterns are incompatible. Therefore at least $12 + \min(18, 14) = 26$ patterns are incompatible.

Case 3.2: There is an incoming edge into v_3 in $E(V)$. By Lemma 3 (ii), all the incoming edges into v_3 are on one side of V, $|V_{1*}| \geq 4$, $|V_{2*}| \geq 2$ and $|V_{3*}| = 2$.

Case 3.2.1: The incoming edges into v_2 in $E(V)$ are on both sides of V.

If the incoming edges into v_3 w.r.t. V are in $E^+(V)$, see Fig. 14 (left), then by planarity $u_3v_3 \notin E(UV)$. So at least $|U_{*3}||V_{3*}| \geq 4 \times 2 = 8$ patterns are incompatible. Also u_2v_1 and u_4v_3 can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{3*}|) = \min(2 \times 4, 6 \times 2) = 8$ patterns are incompatible. By the same token, an outgoing edge from u_4 w.r.t. UV and the edges u_2v_3 and u_3v_2 cannot exist together in $E(UV)$. Therefore at least

$$\min(|U_{*4}||\{\emptyset\}|, |U_{*2}||V_{3*}|, |U_{*3}||V_{2*}|) = \min(6 \times 1, 2 \times 2, 4 \times 2) = 4$$

patterns are incompatible. Similarly, by planarity, an incoming edge into v_1 w.r.t. UV and the edges u_3v_2 and u_4v_3 cannot coexist in $E(UV)$. Therefore at least

$$\min(|\{\emptyset\}||V_{1*}|, |U_{*3}||V_{2*}|, |U_{*4}||V_{3*}|) = \min(1 \times 4, 4 \times 2, 6 \times 2) = 4$$

patterns are incompatible. Hence at least $8 + 8 + 4 + 4 = 24$ patterns are incompatible.

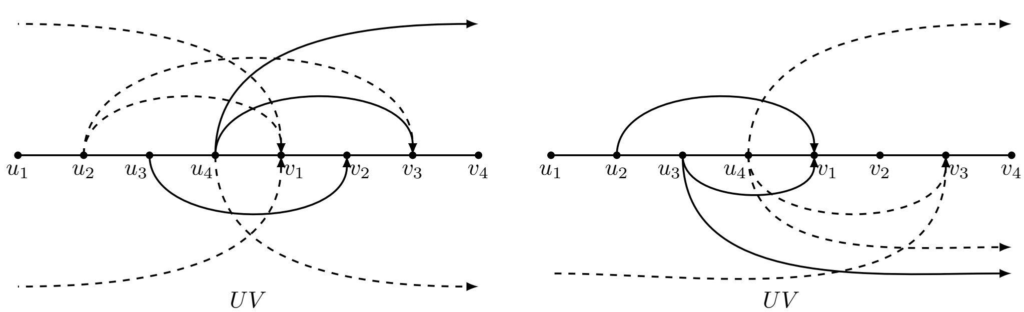

Figure 14: Left: incoming edges into v_2 are on both sides and incoming edges into v_3 are in $E^+(V)$. Right: incoming edges into v_2 are on both sides and incoming edges into v_3 are in $E^-(V)$.

If the incoming edges into v_3 w.r.t. V are in $E^-(V)$, see Fig. 14 (right), then by planarity $u_2v_3 \notin E(UV)$. So at least $|U_{*2}||V_{3*}| \geq 2 \times 2 = 4$ patterns are incompatible. Also u_3v_1 and u_4v_3 can only be in $E^-(UV)$. By planarity both the edges cannot be in $E(UV)$. Hence at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{3*}|) = \min(4 \times 4, 6 \times 2) = 12$ edges are incompatible. By planarity an outgoing going edge from u_3 w.r.t. UV and an incoming edge into v_3 w.r.t. UV cannot exist together in $E(UV)$. Therefore at least $\min(|U_{*3}||\{\emptyset\}|, |\{\emptyset\}||V_{3*}|) = \min(4 \times 1, 1 \times 2) = 2$ patterns are incompatible. By the same token, an outgoing edge from u_4 w.r.t. UV and the edges u_2v_1 and u_3v_1 cannot be together in $E(UV)$. Hence at least $\min(|U_{*2}||V_{1*}|, |U_{*3}||V_{1*}|, |U_{*4}||\{\emptyset\}|) = \min(2 \times 4, 4 \times 4, 6 \times 1) = 6$ patterns are incompatible. So at least $4 + 12 + 2 + 6 = 24$ patterns are incompatible.

Case 3.2.2: All the incoming edges into v_2 in $E(V)$ are on one side of V and all the incoming edges into v_2 and v_3 in $E(V)$ are on the same side of V.

If the incoming edges into v_2 and v_3 w.r.t. V are in $E^+(V)$, see Fig. 15 (left), by planarity u_3v_2 and u_3v_3 are not in $E(UV)$. So at least $|U_{*3}||V_{2*}| + |U_{*3}||V_{3*}| = 4 \times 2 + 4 \times 2 = 16$ patterns are incompatible. Also u_2v_1 and u_4v_2 can only be in $E^+(UV)$. By planarity both the edges cannot be in $E(UV)$. Hence at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(2 \times 4, 6 \times 2) = 8$ patterns are incompatible. Therefore at least $16 + 8 = 24$ patterns are incompatible.

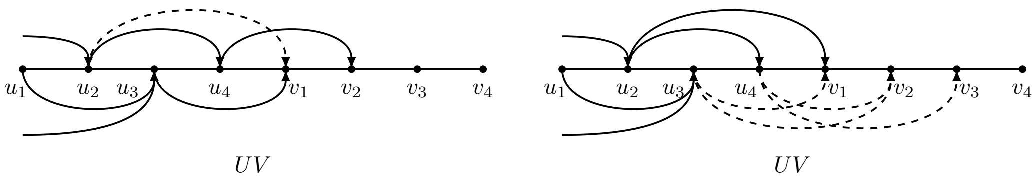

Figure 15: Left: incoming edges into v_2, v_3 are in $E^+(V)$. Right: incoming edges into v_2, v_3 are in $E^-(V)$.

If the incoming edges into v_2 and v_3 w.r.t. V are in $E^-(V)$, see Fig. 15 (right), by planarity $u_2v_3 \notin E(UV)$. So at least $|U_{*2}||V_{3*}| \geq 2 \times 2 = 4$ patterns are incompatible. Also u_3v_1, u_3v_2 u_4v_2, u_4v_3 can only be in $E^-(UV)$. By planarity either u_3v_1 or u_4v_2 and either u_3v_2 or u_4v_3 can be in $E(UV)$. Hence at least

$$\begin{aligned}&\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) + \min(|U_{*3}||V_{2*}|, |U_{*4}||V_{3*}|)\\ =&\min(4 \times 4, 6 \times 2) + \min(4 \times 2, 6 \times 2) = 20\end{aligned}$$

combinations are incompatible. Therefore at least $4 + 20 = 24$ patterns are incompatible.

Case 3.2.3: All the incoming edges into v_2 are on one side of V and all the incoming edges into v_3 are on the opposite side of V. Let the incoming edges w.r.t. V in $E^+(V)$ are into v_i and the incoming

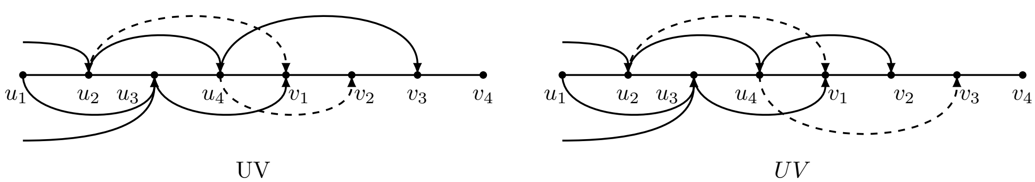

Figure 16: Left: incoming edge into v_2 is in $E^-(V)$ and into v_3 is in $E^+(V)$. Right: Incoming edge into v_2 is in $E^+(V)$ and into v_3 is in $E^-(V)$.

edges w.r.t. V in $E^-(V)$ are into v_j. So either $i = 2$, $j = 3$ or $i = 3$, $j = 2$, see Fig. 16. Therefore $|V_{i*}|$, $|V_{j*}|$ are at least $\min(|V_{2*}|, |V_{3*}|) = 2$. By planarity $u_3v_i \notin E(UV)$. So at least $|U_{*3}||V_{i*}| \geq 4 \times 2 = 8$ patterns are incompatible. Also u_2v_1 and u_4v_i can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{i*}|) = \min(2 \times 4, 6 \times 2) = 8$ patterns are incompatible. Similarly both u_3v_1 and u_4v_j can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{j*}|) = \min(4 \times 4, 6 \times 2) = 12$ patterns are incompatible. Therefore at least $8 + 8 + 12 = 28$ patterns are incompatible. □

Lemma 8. *Consider a group UV consisting of two consecutive groups of 4 vertices, where $|I(U)| \geq 12$ and $|I(V)| = 12$. Then UV allows at most 120 incidence patterns.*

Proof. We distinguish two cases depending on $|I(U)|$.

Case 1: $|I(U)| = 12$. Then by Lemma 4 (i) & (ii), for each vertex u_i, all the outgoing edges from u_i, if any, are on one side of U. Since $|I(UV)| \leq |I(U)| \cdot |I(V)| = 12 \times 12 = 144$, it suffices to show that at least $144 - 120 = 24$ of these patterns are incompatible. We distinguish three cases depending on which vertex in U have outgoing edges and which sides are containing those outgoing edges.

Case 1.1: U has outgoing edges from exactly one vertex. By Lemma 4 (ii), they are from u_3 and we have $|U_{*3}| = 4$ and $|U_{*4}| = 7$. For simplicity assume they are in $E^-(U)$.

Case 1.1.1: V has incoming edges into exactly one vertex. By Lemma 4 (iv), they are into v_2 and we have $|V_{2*}| = 4$ and $|V_{1*}| = 7$.

If the incoming edges into v_2 w.r.t. V are in $E^-(V)$, see Fig. 17 (left), then u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(4 \times 7, 7 \times 4) = 28$ patterns are incompatible.

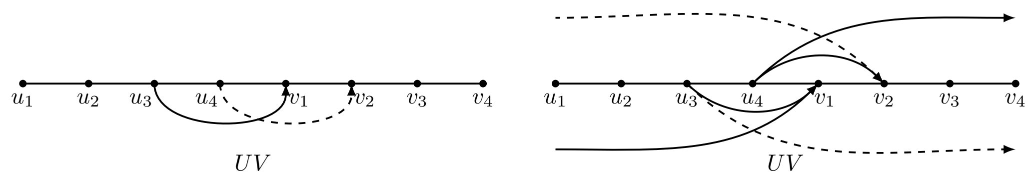

Figure 17: Left: the incoming edges into v_2 w.r.t. V are in $E^-(V)$. Right: the incoming edges into v_2 w.r.t. V are in $E^+(V)$.

If the incoming edges into v_2 w.r.t. V are in $E^+(V)$, see Fig. 17 (right), then by planarity u_3v_2 is not in $E(UV)$. So $|U_{*3}||V_{2*}| = 4 \times 4 = 16$ patterns are incompatible. By planarity the edge u_3v_1, an outgoing edge from u_4 w.r.t. UV and an incoming edge into v_2 w.r.t. UV cannot be in $E(UV)$ together. Therefore at least

$$\min(|U_{*3}||V_{1*}|, |U_{*4}||\{\emptyset\}|, |\{\emptyset\}||V_{2*}|) = \min(4 \times 7, 7 \times 1, 1 \times 4) = 4$$

patterns are incompatible. By the same token, the edge u_4v_2, an incoming edge into v_1 w.r.t. UV and an outgoing edge from u_3 w.r.t. UV cannot coexist in $E(UV)$. Therefore at least

$$\min(|U_{*4}||V_{2*}|, |\{\emptyset\}||V_{1*}|, |U_{*3}||\{\emptyset\}|) = \min(7 \times 4, 1 \times 7, 4 \times 1) = 4$$

patterns are incompatible. So $16 + 4 + 4 = 24$ patterns are incompatible. Observe that this group, UV, has 120 patterns, which is the maximum number of patterns.

Case 1.1.2: V has incoming edges into more than one vertex. Then by Lemma 4 (iv), there are incoming edges into v_3 and v_2 and we have $|V_{3*}| = 2$, $|V_{2*}| \geq 3$ and $|V_{1*}| \geq 5$. We distinguish four scenarios based on which sides of V are containing the incoming edges into v_2 and v_3.

If the incoming edges into v_2 and v_3 w.r.t. V are in $E^-(V)$, see Fig. 18 (left), then both u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both the edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(4 \times 5, 7 \times 3) = 20$ patterns are incompatible. By the same token, both u_3v_2 and u_4v_3 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{2*}|, |U_{*4}||V_{3*}|) = \min(4 \times 3, 7 \times 2) = 12$ other patterns are incompatible. Overall, at least $20 + 12 = 32$ patterns are incompatible.

If the incoming edges into v_2 and v_3 w.r.t. V are in $E^+(V)$, see Fig. 18 (right), then by planarity both u_3v_2 and u_3v_3 are not in $E(UV)$. So at least $|U_{*3}||V_{2*}| + |U_{*3}||V_{3*}| = 4 \times 3 + 4 \times 2 = 20$

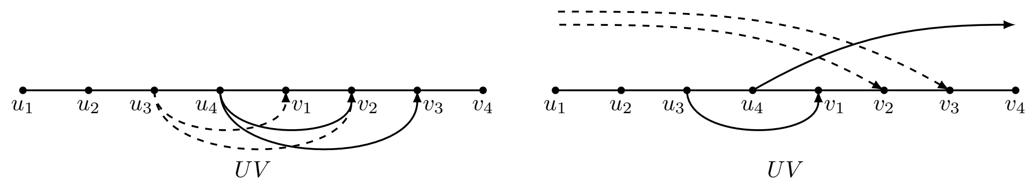

Figure 18: Left: the incoming edges into v_2 and v_3 w.r.t. V are in $E^-(V)$. Right: the incoming edges into v_2 and v_3 w.r.t. V are in $E^+(V)$.

patterns are incompatible. By planarity incoming edges into v_2 and v_3 w.r.t. UV cannot coexist with outgoing edges from u_4 w.r.t. U and the edge u_3v_1. So at least

$$\min(|\{\emptyset\}||V_{2*}| + |\{\emptyset\}||V_{3*}|, |U_{*4}||\{\emptyset\}|, |U_{*3}||V_{1*}|) = \min(1 \times 3 + 1 \times 2, 7 \times 1, 4 \times 5) = 5$$

patterns are incompatible. So at least $20 + 5 = 25$ patterns are incompatible.

If the incoming edges into v_2 w.r.t. V are in $E^-(V)$ and the incoming edges into v_3 w.r.t. V are in $E^+(V)$, see Fig. 19 (left), then by planarity u_3v_3 is not in $E(UV)$. So at least $|U_{*3}||V_{3*}| = 4 \times 2 = 8$ patterns are incompatible. Also both u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(4 \times 5, 7 \times 3) = 20$ patterns are incompatible. So at least $8 + 20 = 28$ patterns are incompatible.

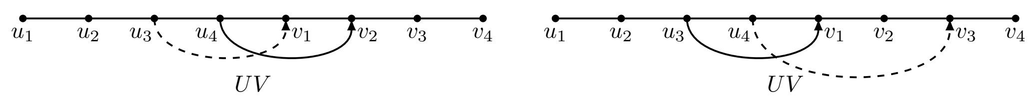

Figure 19: Left: the incoming edges into v_2 w.r.t. V are in $E^-(V)$. Right: the incoming edges into v_2 w.r.t. V are in $E^+(V)$.

If the incoming edges into v_2 w.r.t. V are in $E^+(V)$ and the incoming edges into v_3 w.r.t. V are in $E^-(V)$, see Fig. 19 (right), then by planarity u_3v_2 is not in $E(UV)$. So at least $|U_{*3}||V_{2*}| = 4 \times 3 = 12$ patterns are incompatible. Also, both u_3v_1 and u_4v_3 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{3*}|) = \min(4 \times 5, 7 \times 2) = 14$ patterns are incompatible. So at least $12 + 14 = 26$ patterns are incompatible.

Case 1.2: U has outgoing edges from u_2 and u_3 and both are on the same side. By Lemma 4 (ii), $|U_{*2}| = 2$, $|U_{*3}| \geq 3$ and $|U_{*4}| \geq 5$. For simplicity assume that the outgoing edges are in $E^-(U)$.

Case 1.2.1: V has incoming edges into exactly one vertex. By Lemma 4 (iv), these are into v_2 and we have $|V_{2*}| = 4$ and $|V_{1*}| = 7$.

If the incoming edges into v_2 are in $E^-(V)$, see Fig. 20 (left), then both u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(3 \times 7, 5 \times 4) = 20$ patterns are incompatible. By planarity an outgoing edge from u_4 w.r.t. UV, an incoming edges into v_1 w.r.t. UV and the edge u_2v_2 cannot coexist in $E(UV)$. So at least

$$\min(|U_{*4}||\{\emptyset\}|, |\{\emptyset\}||V_{1*}|, |U_{*2}||V_{2*}|) = \min(5 \times 1, 1 \times 7, 2 \times 4) = 5$$

patterns are incompatible. So in total at least $20 + 5 = 25$ incidence patterns are incompatible.

If the incoming edges into v_2 are in $E^+(V)$, see Fig. 20 (right), then by planarity u_2v_2 and u_3v_2 are not in $E(UV)$. So at least $|U_{*2}||V_{2*}| + |U_{*3}||V_{2*}| = 2 \times 4 + 3 \times 4 = 20$ patterns are incompatible.

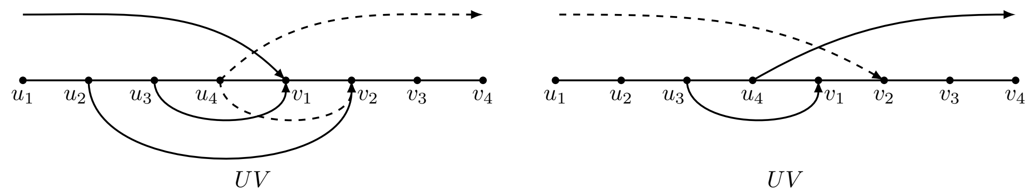

Figure 20: Left: the incoming edges into v_2 w.r.t. V are in $E^-(V)$. Right: the incoming edges into v_2 w.r.t. V are in $E^+(V)$.

By planarity an outgoing edge from u_4 w.r.t. UV, an incoming edges into v_2 w.r.t. UV and the edge u_3v_1 cannot exist together in $E(UV)$. So at least

$$\min(|U_{*4}||\{\emptyset\}|, |\{\emptyset\}||V_{2*}|, |U_{*3}||V_{1*}|) = \min(5 \times 1, 1 \times 4, 3 \times 7) = 4$$

patterns are incompatible. So in total at least $20 + 4 = 24$ incidence patterns are incompatible.

Case 1.2.2: V has incoming edges into more than one vertex. Then by Lemma 4 (iv), there are incoming edges into v_3 and v_2 and we have $|V_{3*}| = 2$, $|V_{2*}| \geq 3$ and $|V_{1*}| \geq 5$. We distinguish four scenarios based on which sides of V are containing the incoming edges into v_2 and v_3.

If the incoming edges into v_2 and v_3 w.r.t. V are in $E^-(V)$, see Fig. 21 (left), then both u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both the edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(3 \times 5, 5 \times 3) = 15$ patterns are incompatible. By the same token, both u_2v_1 and u_4v_3 can only be in $E^-(UV)$. By planarity both the edges cannot be in $E(UV)$. So at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{3*}|) = \min(2 \times 5, 5 \times 2) = 10$ patterns are incompatible. So at least $15 + 10 = 25$ patterns are incompatible.

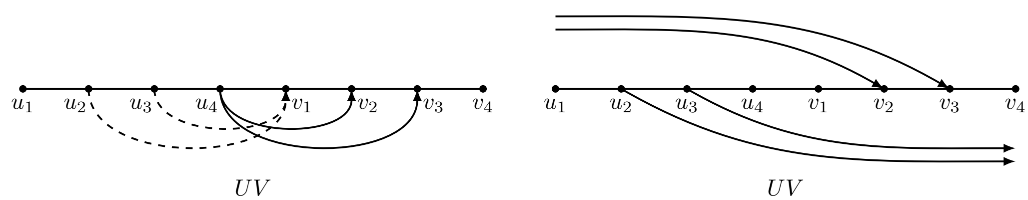

Figure 21: Left: the incoming edges into v_2 and v_3 w.r.t. V are in $E^-(V)$. Right: the incoming edges into v_2 and v_3 w.r.t. V are in $E^+(V)$.

If the incoming edges into v_2 and v_3 w.r.t. V are in $E^+(V)$, see Fig. 21 (right), then by planarity u_2v_2, u_2v_3, u_3v_2 and u_3v_3 are not in $E(UV)$. So at least

$$|U_{*2}||V_{2*}| + |U_{*2}||V_{3*}| + |U_{*3}||V_{2*}| + |U_{*3}||V_{3*}| = 2 \times 3 + 2 \times 2 + 3 \times 3 + 3 \times 2 = 25$$

patterns are incompatible.

If the incoming edges into v_2 w.r.t. V are in $E^-(V)$ and the incoming edges into v_3 w.r.t. V are in $E^+(V)$, see Fig. 22 (left), then by planarity u_2v_3 and u_3v_3 are not in $E(UV)$. So at least $|U_{*2}||V_{3*}|+|U_{*3}||V_{3*}| = 2\times2+3\times2 = 10$ patterns are incompatible. Both u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(3\times5, 5\times3) = 15$ patterns are incompatible. So at least $10+15 = 25$ patterns are incompatible.

If the incoming edges into v_2 w.r.t. V are in $E^+(V)$ and the incoming edges into v_3 w.r.t. V are in $E^-(V)$, see Fig. 22 (right), then by planarity u_2v_2 and u_3v_2 are not in $E(UV)$. So at least $|U_{*2}||V_{2*}| + |U_{*3}||V_{2*}| = 2 \times 3 + 3 \times 3 = 15$ patterns are incompatible. Both u_3v_1 and

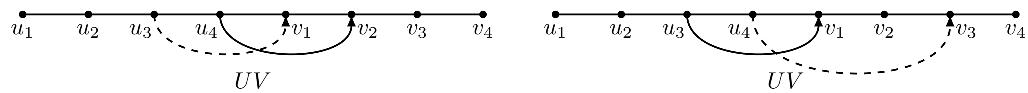

Figure 22: Left: the incoming edges into v_2 w.r.t. V are in $E^-(V)$. Right: the incoming edges into v_2 w.r.t. V are in $E^+(V)$.

u_4v_3 can only be in $E^-(UV)$. By planarity both the edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{3*}|) = \min(3\times 5, 5\times 2) = 10$ patterns are incompatible. So at least $15+10 = 25$ patterns are incompatible.

Case 1.3: U has outgoing edges from u_2 and u_3 and both are on opposite sides. By Lemma 4 (ii), $|U_{*2}| = 2$, $|U_{*3}| \geq 3$ and $|U_{*4}| \geq 5$. For simplicity assume that the outgoing edges from u_3 w.r.t. U are in $E^-(U)$.

Case 1.3.1: V has incoming edges into exactly one vertex. By Lemma 4 (iv), they are into v_2 and we have $|V_{2*}| = 4$ and $|V_{1*}| = 7$.

If the incoming edges into v_2 w.r.t. V are in $E^-(V)$, see Fig. 23 (left), then by planarity u_2v_2 is not in $E(UV)$. So at least $|U_{*2}||V_{2*}| = 2 \times 4 = 8$ patterns are incompatible. Also both u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(3\times 7, 5\times 4) = 20$ patterns are incompatible. So at least $8+20 = 28$ patterns are incompatible.

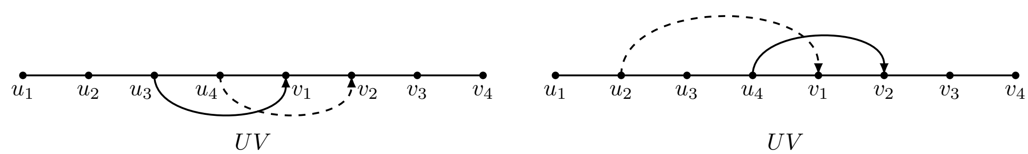

Figure 23: Left: the incoming edges into v_2 w.r.t. V are in $E^-(V)$. Right: the incoming edges into v_2 w.r.t. V are in $E^+(V)$.

If the incoming edges into v_2 w.r.t. V are in $E^+(V)$, see Fig. 23 (right), then by planarity u_3v_2 is not in $E(UV)$. So at least $|U_{*3}||V_{2*}| = 3 \times 4 = 12$ patterns are incompatible. Also both u_2v_1 and u_4v_2 can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(2 \times 7, 5 \times 4) = 14$ patterns are incompatible. So at least $12 + 14 = 26$ patterns are incompatible.

Case 1.3.2: V has incoming edges into more than one vertex. Then by Lemma 4 (iv), there are incoming edges into v_3 and v_2 and we have $|V_{3*}| = 2$, $|V_{2*}| \geq 3$ and $|V_{1*}| \geq 5$. We distinguish four scenarios based on which sides of V are containing the incoming edges into v_2 and v_3.

If the incoming edges into v_2 and v_3 w.r.t. V are in $E^-(V)$, see Fig. 24 (left), then by planarity u_2v_2 and u_2v_3 are not in $E(UV)$. So at least $|U_{*2}||V_{2*}|+|U_{*2}||V_{3*}| = 2\times 3+2\times 2 = 10$ patterns are incompatible. Also both u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(3\times 5, 5\times 3) = 15$ patterns are incompatible. So at least $10 + 15 = 25$ patterns are incompatible.

If the incoming edges into v_2 and v_3 w.r.t. V are in $E^+(V)$, see Fig. 24 (right), then by planarity u_3v_2 and u_3v_3 are not in $E(UV)$. So at least $|U_{*3}||V_{2*}|+|U_{*3}||V_{3*}| = 3\times 3+3\times 2 = 15$ patterns are incompatible. Also both u_2v_1 and u_4v_2 can only be in $E^+(UV)$. By planarity both edges cannot be

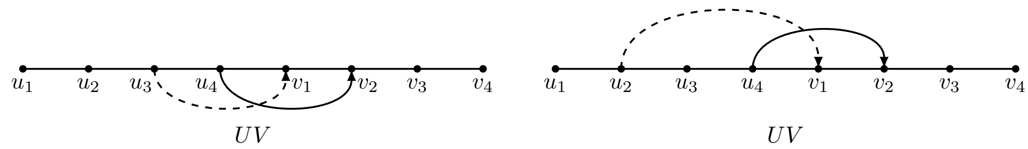

Figure 24: Left: the incoming edges into v_2 and v_3 w.r.t. V are in $E^-(V)$. Right: the incoming edges into v_2 and v_3 w.r.t. V are in $E^+(V)$.

in $E(UV)$. So at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(2\times 5, 5\times 3) = 10$ patterns are incompatible. So at least $15 + 10 = 25$ patterns are incompatible.

If the incoming edges into v_2 w.r.t. V is in $E^-(V)$ and the incoming edges into v_3 w.r.t. V are in $E^+(V)$, see Fig. 25 (left), then both u_2v_1 and u_4v_3 can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{3*}|) = \min(2 \times 5, 5 \times 2) = 10$ patterns are incompatible. By similar token both u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(3 \times 5, 5 \times 3) = 15$ patterns are incompatible. So at least $10 + 15 = 25$ patterns are incompatible.

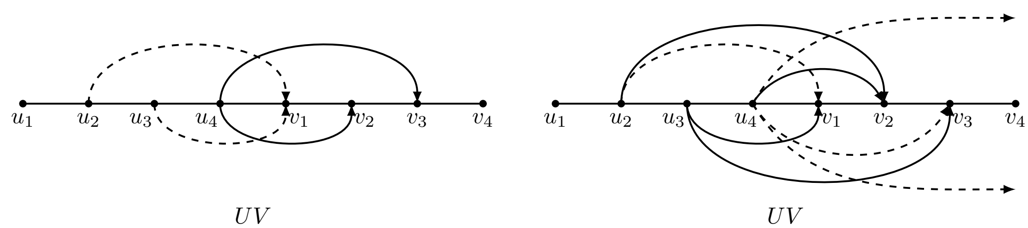

Figure 25: Left: the incoming edges into v_2 w.r.t. V are in $E^-(V)$. Right: the incoming edges into v_2 w.r.t. V are in $E^+(V)$.

If the incoming edges into v_2 w.r.t. V is in $E^+(V)$ and the incoming edges into v_3 w.r.t. V are in $E^-(V)$, see Fig. 25 (right), then both u_2v_1 and u_4v_2 can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(2 \times 5, 5 \times 3) = 10$ patterns are incompatible. Also both u_3v_1 and u_4v_3 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{3*}|) = \min(3 \times 5, 5 \times 2) = 10$ patterns are incompatible. By planarity an outgoing edge from u_4 w.r.t. UV, the edge u_2v_2 and the edge u_3v_3 cannot exist together in $E(UV)$. So at least

$$\min(|U_{*4}||\{\emptyset\}|, |U_{*2}||V_{2*}|, |U_{*3}||V_{3*}|) = \min(5 \times 1, 2 \times 3, 3 \times 2) = 5$$

patterns are incompatible. So in total at least $10+10+5 = 25$ incidence patterns are incompatible.

Case 2: $|I(U)| = 13$. By Lemma 5, U is either A or A^R. If U is A^R, then after reflecting UV about the horizontal axis, U is A. We analyze the cases based on this assumption. Since $|I(UV)| \le |I(U)| \cdot |I(V)| = 13 \times 12 = 156$, it suffices to show that at least $156 - 120 = 36$ of these patterns are incompatible.

Case 2.1: V has incoming edges into exactly one vertex. Then by Lemma 4 (iv), they are into v_2 and we have $|V_{2*}| = 4$ and $|V_{1*}| = 7$.

If the incoming edges into v_2 w.r.t. V are in $E^-(V)$, see Fig. 26 (left), then by planarity u_2v_2 is not in $E(UV)$. So at least $|U_{*2}||V_{2*}| = 2 \times 4 = 8$ patterns are incompatible. Both u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least

$\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(4 \times 7, 6 \times 4) = 24$ patterns are incompatible. By planarity an outgoing edge from u_4 w.r.t. UV, the edge u_2v_1 and the edge u_3v_2 cannot exist together in $E(UV)$. So at least

$$\min(|U_{*4}||\{\emptyset\}|, |U_{*2}||V_{1*}|, |U_{*3}||V_{2*}|) = \min(6 \times 1, 2 \times 7, 4 \times 4) = 6$$

patterns are incompatible. So in total at least $8 + 24 + 6 = 38$ incidence patterns are incompatible.

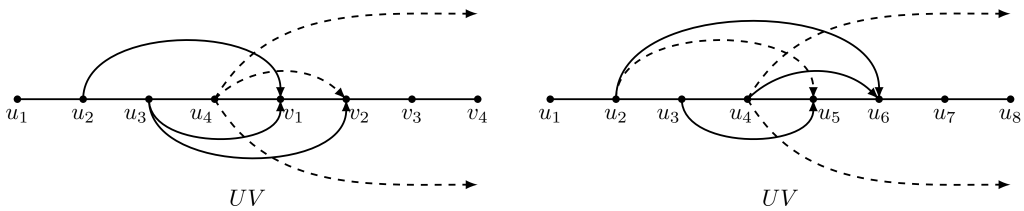

Figure 26: Left: the incoming edges into v_2 w.r.t. V are in $E^-(V)$. Right: the incoming edges into v_2 w.r.t. V are in $E^+(V)$.

If the incoming edges into v_2 w.r.t. V are in $E^+(V)$, see Fig. 26 (right), then by planarity u_3v_2 is not allowed. So at least $|U_{*3}||V_{2*}| = 4 \times 4 = 16$ patterns are incompatible. Both u_2v_1 and u_4v_2 can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(2 \times 7, 6 \times 4) = 14$ patterns are incompatible. By planarity an outgoing edge from u_4 w.r.t. UV, the edge u_2v_2 and the edge u_3v_1 cannot exist together in $E(UV)$. So at least

$$\min(|U_{*4}||\{\emptyset\}|, |U_{*2}||V_{2*}|, |U_{*3}||V_{1*}|) = \min(6 \times 1, 2 \times 4, 4 \times 7) = 6$$

patterns are incompatible. So at least $16 + 14 + 6 = 36$ patterns are incompatible.

Case 2.2: V has incoming edges into more than one vertex. By Lemma 4 (iv), there are incoming edges into v_3 and v_2 and we have $|V_{3*}| = 2$, $|V_{2*}| \geq 3$ and $|V_{1*}| \geq 5$. We distinguish four scenarios based on which sides of V are containing the incoming edges into v_2 and v_3.

If the incoming edges into v_2 and v_3 w.r.t. V are in $E^-(V)$, see Fig. 27 (left), then by planarity u_2v_2 is not in $E(UV)$. So at least $|U_{*2}||V_{2*}| = 2 \times 3 = 6$ patterns are incompatible. Both u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(4 \times 5, 6 \times 3) = 18$ patterns are incompatible. Similarly both u_3v_2 and u_4v_3 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{2*}|, |U_{*4}||V_{3*}|) = \min(4 \times 3, 6 \times 2) = 12$ patterns are incompatible. So at least $6 + 18 + 12 = 36$ patterns are incompatible.

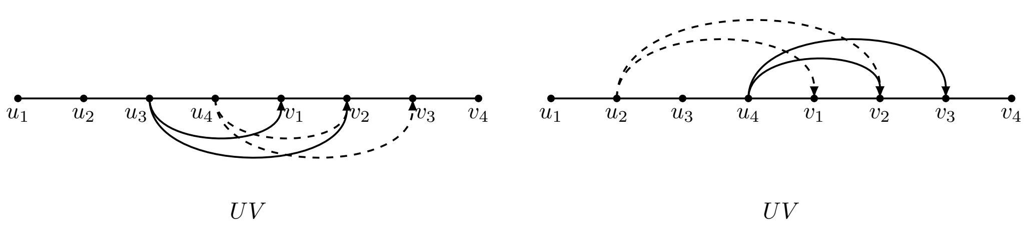

Figure 27: Left: the incoming edges into v_2 and v_3 w.r.t. V are in $E^-(V)$. Right: the incoming edges into v_2 and v_3 w.r.t. V are in $E^+(V)$.

If the incoming edges into v_2 and v_3 w.r.t. V are in $E^+(V)$, see Fig. 27 (right), then by planarity both u_3v_2 and u_3v_3 are not in $E(UV)$. So at least $|U_{*3}||V_{2*}| + |U_{*3}||V_{3*}| = 4 \times 3 + 4 \times 2 = 20$ patterns are incompatible. Also both u_2v_1 and u_4v_2 can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(2 \times 5, 6 \times 3) = 10$ patterns are incompatible. Similarly both u_2v_2 and u_4v_3 can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*2}||V_{2*}|, |U_{*4}||V_{3*}|) = \min(2 \times 3, 6 \times 2) = 6$ patterns are incompatible. So at least $20 + 10 + 6 = 36$ patterns are incompatible.

If the incoming edges into v_2 w.r.t. V is in $E^-(V)$ and the incoming edges into v_3 w.r.t. V are in $E^+(V)$, see Fig. 28 (left), then by planarity u_3v_3 is not in $E(UV)$. So at least $|U_{*3}||V_{3*}| = 4 \times 2 = 8$ patterns are incompatible. Both u_2v_1 and u_4v_3 can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{3*}|) = \min(2 \times 5, 6 \times 2) = 10$ patterns are incompatible. Similarly both u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(4 \times 5, 6 \times 3) = 18$ patterns are incompatible. So at least $8 + 10 + 18 = 36$ patterns are incompatible.

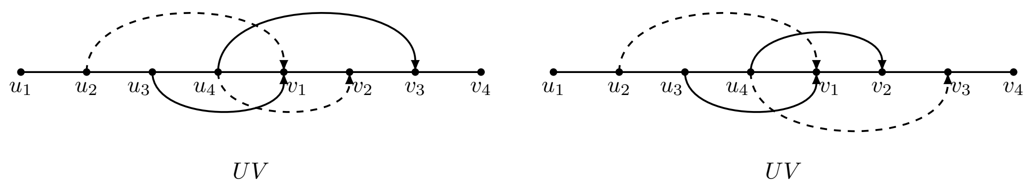

Figure 28: Left: the incoming edges into v_2 w.r.t. V are in $E^-(V)$. Right: the incoming edges into v_2 w.r.t. V are in $E^+(V)$.

If the incoming edges into v_2 w.r.t. V is in $E^+(V)$ and the incoming edges into v_3 w.r.t. V are in $E^-(V)$, see Fig. 28 (right), then by planarity both u_2v_3 and u_3v_2 are not in $E(UV)$. So at least $|U_{*2}||V_{3*}|+|U_{*3}||V_{2*}| = 2\times2+4\times3 = 16$ patterns are incompatible. Both u_2v_1 and u_4v_2 can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(2 \times 5, 6 \times 3) = 10$ patterns are incompatible. Similarly both u_3v_1 and u_4v_3 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{3*}|) = \min(4 \times 5, 6 \times 2) = 12$ patterns are incompatible. So at least $16 + 10 + 12 = 38$ patterns are incompatible. ☐

Lemma 9. *Consider a group UV consisting of two consecutive groups of 4 vertices, where $|I(U)| = |I(V)| = 13$. Then UV allows at most 120 incidence patterns.*

Proof. By Lemma 5, U is either A or A^R. We may assume, after reflecting UV about a horizontal axis, that U is A. Therefore $|U_{*2}| = 2$, $|U_{*3}| = 4$ and $|U_{*4}| = 6$, see Figure 6. Similarly, Lemma 5 implies that V is either A or A^R. We distinguish two cases depending on whether V is A or A^R. The cross product of $I(U)$ and $I(V)$ yields $13 \times 13 = 169$ possible patterns. It suffices to show that at least $169 - 120 = 49$ of these patterns are incompatible.

Case 1: V is A, see Figure 29 (left). By planarity u_2v_3 and u_3v_2 are not in $E(UV)$. So at least $|U_{*2}||V_{3*}| + |U_{*3}||V_{2*}| = 2 \times 2 + 4 \times 4 = 20$ patterns are incompatible. Further, u_2v_1 and u_4v_2 can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(2 \times 6, 6 \times 4) = 12$ patterns are incompatible. Similarly u_3v_1 and u_4v_3 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Therefore at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{3*}|) = \min(4 \times 6, 6 \times 2) = 12$ patterns are incompatible. By planarity an outgoing edge from u_4 w.r.t. UV and the edges u_2v_2 and u_3v_1 cannot exist together in $E(UV)$. So at least

$$\min(|U_{*4}||\emptyset|, |U_{*2}||V_{2*}|, |U_{*3}||V_{3*}|) = \min(6 \times 1, 2 \times 4, 4 \times 2) = 6$$

patterns are incompatible. Overall, at least $20 + 12 + 12 + 6 = 50$ patterns are incompatible.

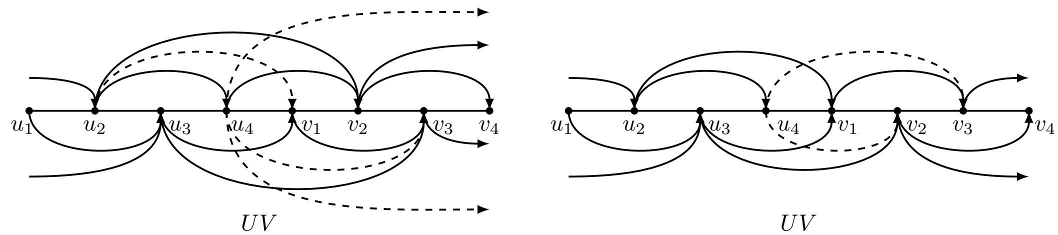

Figure 29: Left: V is A. Right: V is A^R.

Case 2: V is A^R, see Figure 29 (right). By planarity u_2v_2 and u_3v_3 are not in $E(UV)$. So at least $|U_{*2}||V_{2*}| + |U_{*3}||V_{3*}| = 2 \times 4 + 4 \times 2 = 16$ patterns are incompatible. Also u_2v_1 and u_4v_3 can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{*2}||V_{1*}|, |U_{*4}||V_{3*}|) = \min(2 \times 6, 6 \times 2) = 12$ patterns are incompatible. Similarly u_3v_1 and u_4v_2 can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{*3}||V_{1*}|, |U_{*4}||V_{2*}|) = \min(4 \times 6, 6 \times 4) = 24$ patterns are incompatible. Overall, at least $16 + 12 + 24 = 52$ patterns are incompatible. □

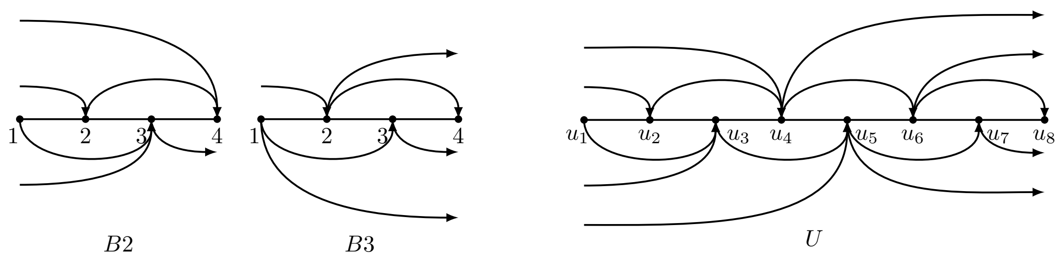

Figure 30: U has 120 patterns. The 24 missing patterns are 123678, 12367, 12368, 1236, 123, 13678, 1367, 1368, 136, 13, 23678, 2367, 2368, 236, 23, 3678, 367, 368, 36, 3, 678, 67, 68, 6.

Lemma 10. *Every group on 8 vertices has at most 120 incidence patterns, and this bound is the best possible. Consequently,* $p_8 = 120$.

Proof. A group of 8, denoted by UV, where U and V are the groups induced by the first and last four vertices of UV, respectively. If $|I(U)| \leq 9$ or $|I(V)| \leq 9$, then $|I(UV)| \leq |I(U)| \cdot |I(V)| \leq 9 \times 13 = 117$ by Lemma 5. Otherwise, Lemmas 6–9 show that $|I(UV)| \leq 120$.

Consider the group $(U, E^-(U), E^+(U))$ of 8 vertices depicted in Fig. 30 (right). The first and second half of U are the groups $B2$ and $B3$ in Fig. 30 (left), each with 12 patterns. Observe that exactly 24 patterns are incompatible, thus U has exactly $|I(B2)| \cdot |I(B3)| - 24 = 12 \times 12 - 24 = 120$ patterns. Aside from reflections, the extremal group of 8 vertices in Fig. 30 (right) is unique. □

5 Groups of 9, 10, and 11 vertices via computer search

The application of the same fingerprinting technique to groups of 9, 10, and 11 vertices via a computer program[1] shows that

- A group of 9 vertices allows at most 201 incidence patterns; the extremal configuration appears in Fig. 31. This yields the upper bound of $O(n^3 201^{n/9}) = O(1.8027^n)$ for the number of

[1]Refer to the .c file and the Appendix within the source at `arXiv:1608.04812`.

monotone paths in an n-vertex triangulation. Aside from reflections, the extremal group of 9 vertices in Fig. 31 (left) is unique.

- A group of 10 vertices allows at most 346 incidence patterns; the extremal configuration appears in Fig. 32. This yields the upper bound of $O(n^3 346^{n/10}) = O(1.7944^n)$ for the number of monotone paths in an n-vertex triangulation, as given in Theorem 1. Aside from reflections, the extremal group of 10 vertices in Fig. 32 is unique.
- A group of 11 vertices allows at most 591 incidence patterns; the extremal configuration appears in Fig. 33. This yields the upper bound of $O(n^3 591^{n/11}) = O(1.7864^n)$ for the number of monotone paths in an n-vertex triangulation. Aside from reflections, the extremal group of 11 vertices in Fig. 33 (left) is unique.

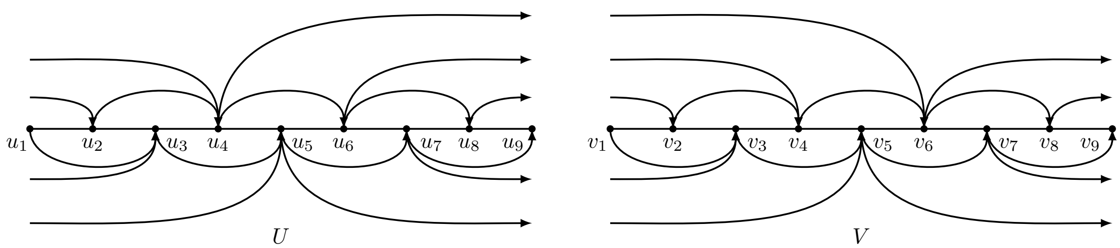

Figure 31: Groups U and V (hence also U^R and V^R) are the only groups of 9 vertices with 201 incidence patterns. Observe that V is the reflection of U in the y-axis.

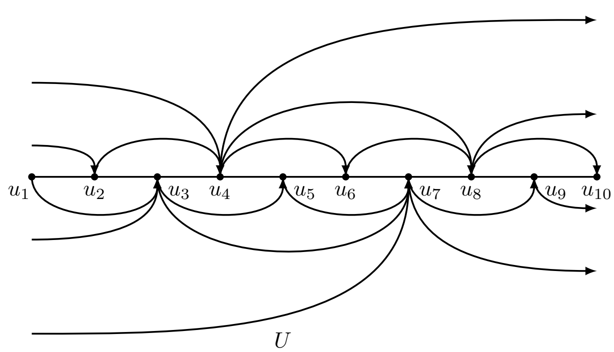

Figure 32: Group U (hence also U^R) is the only group of 10 vertices with 346 incidence patterns. Observe that the reflection of U in the y-axis is U^R.

To generate all groups of k vertices, the program first generates all possible *sides* of k vertices, essentially by brute force. A side of k vertices $V = \{v_1, \dots v_k\}$ is represented by a directed planar graph with $k+2$ vertices, where the edges $v_0 v_i$ and $v_i v_{k+1}$, for $1 \le i \le k$, denote an incoming edge into v_i and an outgoing edge from v_i, respectively. The edge $v_0 v_{k+1}$ represents the $\emptyset$ pattern. Note that $\xi_0 \cup v_0 v_{k+1}$ forms a plane cycle on $k+2$ vertices in the underlying undirected graph. Therefore, $E^+(V)$ and $E^-(V)$ can each have at most $(k+2)-3 = k-1$ edges. After all possible sides are generated, the program combines all pairs of sides with no common inner edge to generate a group $(V, E^-(V), E^+(V))$. For each generated group, the program calculates the corresponding number of patterns and in the end returns the group with the maximum number of patterns.

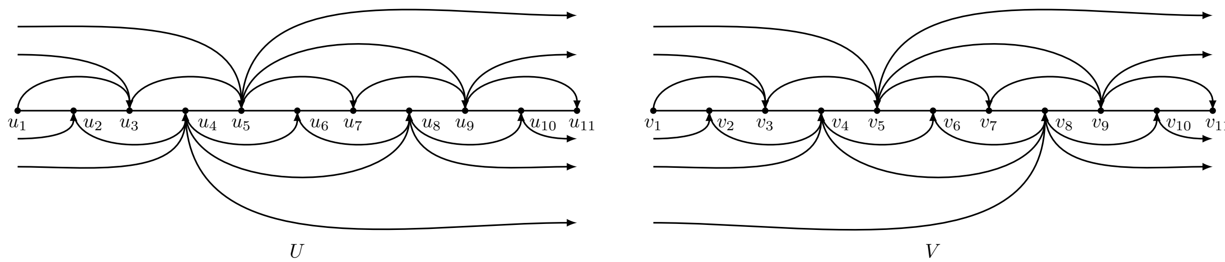

Figure 33: Groups U and V (hence also U^R and V^R) are the only groups of 11 vertices with 591 incidence patterns. Observe that V is the reflection of U in the y-axis.

Remark. It is interesting to observe how the structure of the unique extremal groups of 9, 10 and 11 vertices (depicted in Figures 31, 32 and 33) match that of the current best lower bound construction illustrated in Fig. 1 (right).

6 Counting and enumerating monotone paths

Counting and enumerating x-monotone paths. Let $G = (V, E)$ be a plane geometric graph with n vertices. We first observe that the number of x-monotone paths in G can be computed by a sweep-line algorithm. For every vertex $v \in V$, denote by $m(v)$ the number of (directed) nonempty x-monotone paths that end at v. Sweep a vertical line ℓ from left to right, and whenever ℓ reaches a vertex v, we compute $m(v)$ according to the relation

$$m(v) = \sum_{q \in L(v)} [m(q) + 1],$$

where $L(v)$ denotes the set of neighbors of vertex v in G that lie to the left of v. The total number of x-monotone paths in G is $\sum_{v \in V} m(v)$. For every $v \in V$, the computation of $m(v)$ takes $O(\deg(v))$ time, thus computing $m(v)$ for all $v \in V$ takes $\sum_{v \in V} O(\deg(v)) = O(n)$ time. The algorithm for computing the number of x-monotone paths in G takes $O(n)$ time if the vertices are in sorted order, or $O(n \log n)$ time otherwise.

The sweep-line algorithm can be adapted to enumerate the x-monotone paths in G in $O(n \log n + K)$ time, where K is the sum of the lengths of all x-monotone paths. For every vertex $v \in V$, denote by $M(v)$ the set of (directed) nonempty x-monotone paths that end at v. When the vertical sweep-line ℓ reaches a vertex v, we compute $M(v)$ according to the relation

$$M(v) = \bigcup_{q \in L(v)} \{(q, v)\} \cup \{p \oplus (q, v) : p \in M(q)\},$$

where the concatenation of two paths, p_1 and p_2, is denoted by $p_1 \oplus p_2$. The set of all x-monotone paths is $\bigcup_{v \in V} M(v)$.

The number (resp., set) of $\mathbf{u}$-monotone paths in G for any direction $\mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ can be computed in a similar way, using the counts $m_{\mathbf{u}}(v)$ (resp., sets $M_{\mathbf{u}}(v)$) and the neighbor sets $L_{\mathbf{u}}(v)$, instead; the overall time per direction is $O(n)$ (resp., $O(n + K_{\mathbf{u}})$) if the vertices are in sorted order, or $O(n \log n)$ (resp., $O(n \log n + K_{\mathbf{u}})$) otherwise.

Computing all monotone paths. For computing the total number of monotone paths over all directions $\mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, some care is required[2]. As shown subsequently, it suffices to consider monotone paths in at most $2|E|$ directions, one direction between any two consecutive directions orthogonal to the edges in E (each edge yields two opposite directions). It is worth noting however, that it does *not* suffice to consider only directions parallel or orthogonal to the edges of G, i.e., the union of monotone paths over all such directions may not contain all monotone paths. Second, observe that once a sufficient set of directions is established, we cannot simply sum up the number of monotone paths over all such directions, since a monotone path in G may be monotone in several of these directions.

We first compute a set of directions that is *sufficient* for our purpose, i.e., every monotone path is monotone with respect to at least one of these directions. We then consider these directions sorted by angle, and for each new direction, we compute the number of *new* paths. In fact, we count *directed* monotone paths, that is, each path will be counted twice, as traversed in two opposite directions. We now proceed with the details.

Partition the edge set E of G into subsets of parallel edges; note that each subset yields two opposite directions. Since $|E| \leq 3n-6$, the edges are partitioned into at most $|E| \leq 3n-6$ subsets. Let $\mathcal{D}$ be a set of direction vectors $\overrightarrow{ab}$ of the edges $(a,b) \in E$, one from each subset of parallel edges. Let $\mathcal{D}^{\perp}$ be a set of vectors obtained by rotating each vector in $\mathcal{D}$ counterclockwise by $\pi/2$ and by $-\pi/2$; note that $|\mathcal{D}^{\perp}| = 2|\mathcal{D}| \leq 2|E| \leq 6n-12$. Sort the vectors in $\mathcal{D}^{\perp}$ by their arguments[3] in cyclic order, and let $\mathcal{U}$ be a set of vector sums of all pairs of consecutive vectors in $\mathcal{D}^{\perp}$; clearly $|\mathcal{U}| \leq |\mathcal{D}^{\perp}| \leq 6n-12$. We show that $\mathcal{U}$ is a sufficient set of directions. Indeed, consider a path ξ monotone with respect to some direction $\mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Observe that the edges in ξ cannot be orthogonal to $\mathbf{u}$; it follows that ξ is still monotone with respect to at least one of the two adjacent directions in $\mathcal{U}$ (closest to $\mathbf{u}$).

We next present the algorithm. Sort the vectors in $\mathcal{U}$ by their arguments, in $O(n \log n)$ time. Let $\mathbf{u}_0 \in \mathcal{U}$ be an vector of minimum argument in $\mathcal{U}$. We first compute the number of $\mathbf{u}_0$-monotone directed paths in G, $\sum_{v \in V} m_{\mathbf{u_0}}(v)$, by the sweep-line algorithm described above in $O(n \log n)$ time.

Consider the directions $\mathbf{u} \in \mathcal{U} \setminus \{\mathbf{u}_0\}$, sorted by increasing arguments. For each $\mathbf{u}$, we maintain the number of directed paths in G that are monotone in some direction between $\mathbf{u}_0$ and $\mathbf{u}$ (implicitly, by the sum of parameters γ defined below). For each new direction $\mathbf{u}$, exactly one subset of parallel edges, denoted $E_{\mathbf{u}}$, becomes $\mathbf{u}$-monotone (i.e., it consists of $\mathbf{u}$-monotone edges). Therefore, it is enough to count the number of $\mathbf{u}$-monotone paths that traverse some edge in $E_{\mathbf{u}}$.

Counting the monotone paths in G. These paths can be counted by sweeping G with a line ℓ orthogonal to $\mathbf{u}$: Sort the vertices in direction $\mathbf{u}$ (ties are broken arbitrarily). Compute two parameters for every vertex $v \in V$:

- the number $m_{\mathbf{u}}(v)$ of $\mathbf{u}$-monotone paths that end at v,
- the number $\gamma_{\mathbf{u}}(v)$ of $\mathbf{u}$-monotone paths that end at v *and* contain some edge from $E_{\mathbf{u}}$.

When reaching vertex v, the sweep-line algorithm computes the first parameter $m_{\mathbf{u}}(v)$ according to the relation:

$$m_{\mathbf{u}}(v) = \sum_{q \in L_{\mathbf{u}}(v)} [m_{\mathbf{u}}(q) + 1]. \tag{2}$$

[2]The algorithm has been revised, as some of the ideas were implemented incorrectly in the earlier conference version.

[3]The *argument* of a vector $\mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is the angle measure in $[0, 2\pi)$ of the minimum counterclockwise rotation that carries the positive x-axis to the ray spanned by $\mathbf{u}$.

The second parameter $\gamma_{\mathbf{u}}(v)$ is computed as follows:

$$\gamma_{\mathbf{u}}(v) = \begin{cases} 1 + m_{\mathbf{u}}(a) + \sum_{q \in L_{\mathbf{u}}(v) \setminus \{a\}} \gamma_{\mathbf{u}}(q) & \text{if } \exists (a,v) \in E_{\mathbf{u}} \text{ with } \langle \overrightarrow{av}, \mathbf{u} \rangle > 0, \\ \sum_{q \in L_{\mathbf{u}}(v)} \gamma_{\mathbf{u}}(q) & \text{otherwise.} \end{cases} \tag{3}$$

The total number of monotone paths, returned by the algorithm in the end is

$$\sum_{v \in V} \left(m_{\mathbf{u_0}}(v) + \sum_{\mathbf{u} \in \mathcal{U} \setminus \{\mathbf{u_0}\}} \gamma_{\mathbf{u}}(v) \right).$$

We next show that the sorted order of vertices in the $O(n)$ directions in $\mathcal{U}$ can be computed in $O(n^2)$ time. Consider the duality transform, where every point $v = (a,b)$ is mapped to a dual line $v^* : y = ax - b$, and every line $\ell : y = ax - b$ is mapped to a dual point $\ell^* = (a,b)$. It is known that the duality preserves the above-below relationship between points and lines [3, Ch. 8]. Note that the lines of slope a are mapped to dual points on the vertical line $x = a$. Consequently, when we sweep V by a line of slope a in direction $\mathbf{u} = (-a, 1)$, we encounter the points in V in the order determined by y-coordinates of the intersections of the vertical line $x = a$ with the dual lines in $V^* = \{v^* : v \in V\}$.

Let $\mathcal{A}$ be the arrangement of the n dual lines in V^* plus the $O(n)$ vertical lines corresponding to the slopes of the vectors in $\mathcal{U}$. The arrangement $\mathcal{A}$ of these $O(n)$ lines has $O(n^2)$ vertices, and can be computed in $O(n^2)$ time [3, Ch. 8]. By tracing the vertical line corresponding to each vector $\mathbf{u} \in \mathcal{U}$ in $\mathcal{A}$, we find its intersection points with the dual lines in V^*, sorted by y-coordinates, in $O(n)$ time. Since $|\mathcal{U}| = O(n)$, the total running time of the algorithm is $O(n^2)$, as claimed.

Enumerating the monotone paths in G. To this end we adapt the formulae (2) and (3) to sets of monotone paths in a straightforward manner. For every vertex $v \in V$, we compute two sets:

- the set $M_{\mathbf{u}}(v)$ of $\mathbf{u}$-monotone paths that end at v,
- the set $\Gamma_{\mathbf{u}}(v)$ of $\mathbf{u}$-monotone paths that end at v *and* contain some edge from $E_{\mathbf{u}}$.

When reaching vertex v, the sweep-line algorithm computes $M_{\mathbf{u}}(v)$ according to the relation:

$$M_{\mathbf{u}}(v) = \bigcup_{q \in L_{\mathbf{u}}(v)} \{(q,v)\} \cup \{p \oplus (q,v) : p \in M(q)\}. \tag{4}$$

The set $\Gamma_{\mathbf{u}}(v)$ is computed as follows:

$$\Gamma_{\mathbf{u}}(v) = \begin{cases} \{(a,v)\} \cup \{p \oplus (a,v) : p \in M_{\mathbf{u}}(a)\} \cup & \\ \qquad \bigcup_{q \in L_{\mathbf{u}}(v) \setminus \{a\}} \{p \oplus (q,v) : p \in \Gamma_{\mathbf{u}}(q)\} & \text{if } \exists (a,v) \in E_{\mathbf{u}} \text{ with } \langle \overrightarrow{av}, \mathbf{u} \rangle > 0, \\ \bigcup_{q \in L_{\mathbf{u}}(v)} \{p \oplus (q,v) : p \in \Gamma_{\mathbf{u}}(q)\} & \text{otherwise.} \end{cases} \tag{5}$$

Now the set of directed monotone paths in G is

$$\bigcup_{v \in V} \left(M_{\mathbf{u}_0(v)} \cup \bigcup_{\mathbf{u} \in \mathcal{U} \setminus \{\mathbf{u}_0\}} \Gamma_{\mathbf{u}}(v) \right),$$

where every undirected monotone path appears twice: once in each direction. This completes the proof of Theorem 2. □

7 Concluding remarks

A path is *simple* if it has no repeated vertices; obviously every monotone path is simple. A directed polygonal path $\xi = (v_1, v_2, \ldots, v_t)$ in $\mathbb{R}^d$ is *weakly monotone* if there exists a nonzero vector $\mathbf{u} \in \mathbb{R}^d$ that has a nonnegative inner product with every directed edge of ξ, that is, $\langle \overrightarrow{v_i v_{i+1}}, \mathbf{u} \rangle \geq 0$ for $i = 1, \ldots, t-1$. In many applications such as local search, a weakly monotone path may be as good as a monotone one, since both guarantee that the objective function is nondecreasing.

It therefore appears as a natural problem to find a tight asymptotic bound on the maximum number of weakly monotone simple paths over all plane geometric graphs with n vertices. As for monotone paths, it is easy to see that triangulations maximize the number of such paths. Recall that μ_n denotes the maximum number (over all directions $\mathbf{u}$) of maximal $\mathbf{u}$-monotone paths in an n-vertex triangulation. Let β_n denote the maximum number (over all directions $\mathbf{u}$) of maximal *weakly* $\mathbf{u}$-monotone simple paths in an n-vertex triangulation.

We clearly have $\beta_n \geq \mu_n$, and so $\beta_n = \Omega(1.7003^n)$. However, β_n could in principle grow faster than μ_n. Let $n = 4$ and consider the three vertices and the center of an equilateral triangle, and the unique triangulation of these four points; shown in Fig 34. Observe that: (i) the 5 paths 132, 1342, 142, 1432, and 12 are weakly $\mathbf{u}$-monotone and maximal, where $\mathbf{u} = (1, 0)$ and yield $\beta_4 = 5$; (ii) the 4 paths 143, 142, 12, and 13 are $\mathbf{u}$-monotone and maximal, where $\mathbf{u} = (\cos \pi/6, \sin \pi/6)$ and yield $\mu_4 = 4$; and so $\beta_4 > \mu_4$.

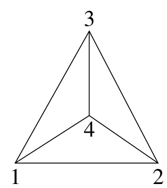

Figure 34: A triangulation of 4 points: the vertices and the center of an equilateral triangle. Note that any two nonadjacent edges are orthogonal.

We conclude with the following open problems.

1. What upper and lower bounds can be derived for β_n? Is $\beta_n = \omega(\mu_n)$?
2. What can be said about counting and enumeration of weakly monotone paths in a given plane geometric graph?

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A Extremal configurations

The groups of 4 vertices with 12 and 11 patterns. There are exactly 4 groups with exactly 12 incidence patterns (modulo reflections about the x-axis); see Fig. 35.

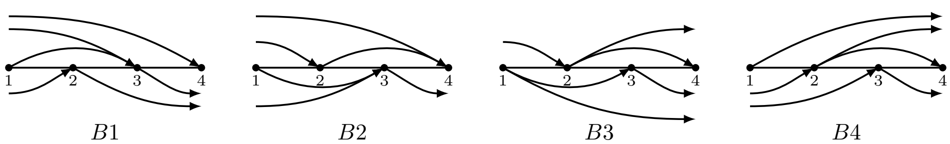


Figure 35: $B1$–$B4$ are the only four groups with 12 incidence patterns.

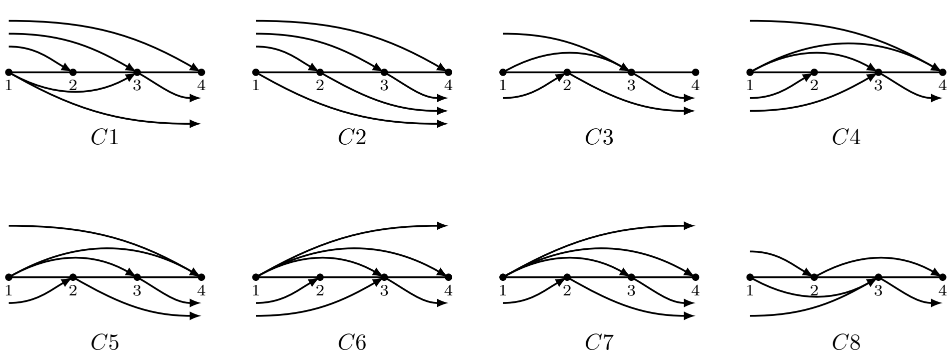
$I(B1)$: $\emptyset, 1234, 123, 12, 134, 13, 234, 23, 2, 34, 3, 4$.

$I(B2)$: $\emptyset, 1234, 123, 124, 134, 13, 234, 23, 24, 34, 3, 4$.

$I(B3)$: $\emptyset, 1234, 123, 124, 12, 134, 13, 1, 23, 234, 24, 2$.

$I(B4)$: $\emptyset, 1234, 123, 124, 12, 1, 23, 234, 24, 2, 34, 3$.

There are exactly 20 groups with exactly 11 incidence patterns (modulo reflections about the x-axis); see Fig. 36.



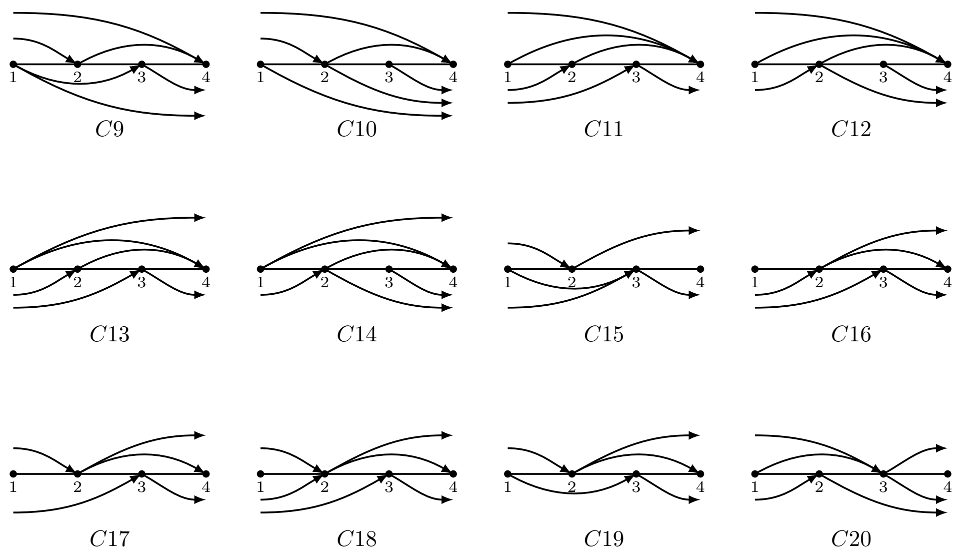


Figure 36: $C1$–$C20$ are the only 20 groups with 11 incidence patterns.

$I(C1) : \emptyset, 1234, 123, 134, 13, 1, 234, 23, 34, 3, 4.$
$I(C2) : \emptyset, 1234, 123, 12, 1, 234, 23, 2, 34, 3, 4.$
$I(C3) : \emptyset, 1234, 123, 12, 134, 13, 234, 23, 2, 34, 3.$
$I(C4) : \emptyset, 1234, 123, 134, 13, 14, 234, 23, 34, 3, 4.$
$I(C5) : \emptyset, 1234, 123, 12, 134, 13, 14, 234, 23, 2, 4.$
$I(C6) : \emptyset, 1234, 123, 134, 13, 14, 1, 234, 23, 34, 3.$
$I(C7) : \emptyset, 1234, 123, 12, 134, 13, 14, 1, 234, 23, 2.$
$I(C8) : \emptyset, 1234, 123, 124, 134, 13, 234, 23, 24, 34, 3.$
$I(C9) : \emptyset, 1234, 123, 124, 134, 13, 1, 234, 23, 24, 4.$
$I(C10) : \emptyset, 1234, 123, 124, 12, 1, 234, 23, 24, 2, 4.$
$I(C11) : \emptyset, 1234, 123, 124, 14, 234, 23, 24, 34, 3, 4.$
$I(C12) : \emptyset, 1234, 123, 124, 12, 14, 234, 23, 24, 2, 4.$
$I(C13) : \emptyset, 1234, 123, 124, 14, 1, 234, 23, 24, 34, 3.$
$I(C14) : \emptyset, 1234, 123, 124, 12, 14, 1, 234, 23, 24, 2.$
$I(C15) : \emptyset, 1234, 123, 12, 134, 13, 234, 23, 2, 34, 3.$
$I(C16) : \emptyset, 1234, 123, 124, 12, 234, 23, 24, 2, 34, 3.$
$I(C17) : \emptyset, 1234, 123, 124, 12, 234, 23, 24, 2, 34, 3.$
$I(C18) : \emptyset, 1234, 123, 124, 12, 234, 23, 24, 2, 34, 3.$
$I(C19) : \emptyset, 1234, 123, 124, 12, 134, 13, 234, 23, 24, 2.$
$I(C20) : \emptyset, 1234, 123, 12, 134, 13, 234, 23, 2, 34, 3.$