Minimum Rectilinear Steiner Tree of n Points in the Unit Square

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Abstract

Chung and Graham conjectured (in 1981) that n points in the unit square $[0,1]^2$ can be connected by a rectilinear Steiner tree of length at most $\sqrt{n}+1$. Here we confirm this conjecture for small values of n, and for some new infinite sequences of values of n (but *not* for all n). As an interesting byproduct we obtain close rational approximations of $\sqrt{n}$ from below, for those n.

Keywords: Minimum rectilinear Steiner tree, integer partition, packing, covering.

1 Introduction

Let S be a finite set of points in the plane. A *Euclidean Steiner tree* (EST) for S is a planar straight line graph spanning S. The *Euclidean Steiner tree problem* asks for the shortest such graph, where the distance between two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. The solutions take form of a tree, that includes all the points in S, called *terminals*, or *sites*, along with possibly some extra vertices, called *Steiner points*. In an optimal solution each Steiner point has degree 3, and any two consecutive incident edges form an $120°$ angle [10]. Obviously, a Euclidean minimum spanning tree (EMST) for a point set can always serve as a possibly suboptimal Euclidean Steiner tree for the same set.

The *rectilinear Steiner tree problem* asks for the shortest Steiner tree where the distance between two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is $|x_1 - x_2| + |y_1 - y_2|$. The solution can be drawn as a *rectilinear Steiner tree* (RST), composed solely of horizontal and vertical edges. The RST problem was first suggested by Hanan [11], who also proved the following structural result regarding optimal solutions. Let $G(S)$ be the grid induced by the point set S by drawing a horizontal and a vertical line through each point of S and retaining only the finite segments between intersection points of these lines (in the axis-aligned bounding box of S). Then there exists a shortest RST for S which uses only segments in $G(S)$ [11]; see also [16, 17].

The following questions were raised by Few [5] in 1955: *What is the greatest length $s(n)$ of a minimum Steiner tree, and what is the greatest length $s_{\llcorner}(n)$ of a minimum rectilinear Steiner tree, of n points in the unit square $[0,1]^2$?*

Few showed that the length of a minimum spanning path of any n points in the unit square is at most $\sqrt{2n}+7/4$ by a constructive proof: lay out about $\sqrt{n}$ equidistant horizontal lines, and then visit the points layer by layer, with the path alternating directions along the horizontal strips. An upper bound with a slightly better multiplicative constant for a path was derived by Karloff [13].

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L. Fejes Tóth [4] had observed earlier that for n points of a regular hexagonal lattice in the unit square, the length of the minimum spanning path is asymptotically equal to $(4/3)^{1/4}\sqrt{n}$, where $(4/3)^{1/4} = 1.0745\ldots$.

By adapting the proof for minimum spanning path, Few also showed that the length of a minimum rectilinear Steiner tree of any n points in the unit square is at most $\sqrt{n} + 7/4$, that is, $s_{\text{L}}(n) \leq \sqrt{n} + 7/4$. Since $s(n) \leq s_{\text{L}}(n)$, this also yields the bound $s(n) \leq \sqrt{n} + 7/4$. Chung and Graham [2] reported an improved upper bound of $s(n) \leq 0.995\sqrt{n}$ (they gave details for an improvement to $0.99995\sqrt{n}$ only) and a lower bound (by the same example of a regular hexagonal lattice in the unit square) of $s(n) \geq (3/4)^{1/4}\sqrt{n} + O(1)$, where $(3/4)^{1/4} = 0.9306\ldots$. Note that $(4/3)^{1/4} = (3/4)^{1/4} \cdot 2/\sqrt{3}$, where $2/\sqrt{3}$ is the conjectured Steiner ratio in the plane.

In every dimension $d \geq 3$, Few [5] showed that the maximum length of a shortest path through n points in the unit cube is $\Theta(n^{1-1/d})$, and that the maximum length of a minimum rectilinear Steiner tree of n points in the unit cube is $O(n^{1-1/d})$. Snyder [18, 19] then proved an asymptotically tight bound of $\Theta(n^{1-1/d})$ for the latter problem, extending the work of Few [5] and Chung and Graham [2]. Among others, some pieces of early work on the geometric variants of the Steiner tree problem (EST or RST) are [3, 6, 7, 8, 12, 20, 21]. Both variants are known to be NP-complete [9]. More recent developments and other problems in geometric networks can be found in [15].

In this paper we study the *rectilinear* version. Chung and Graham [2] observed that $s_{\text{L}}(n) \geq \sqrt{n} + O(1)$ is implied by subsets of points from a suitable square lattice and, in particular, $s_{\text{L}}(k^2) \geq k + 1$ for $k \geq 2$. They also reported the upper bound $s_{\text{L}}(n) \leq \sqrt{n} + 1 + o(1)$, in particular, $s_{\text{L}}(k^2) \leq k + 1$ for $k \geq 2$.

Here we revisit the problem and show that the argument in [2] justifying this upper bound does not stand. Consequently, both the asymptotic upper bound $s_{\text{L}}(n) \leq \sqrt{n} + 1 + o(1)$ and the upper bound $s_{\text{L}}(k^2) \leq k + 1$ for $k \geq 2$ remain without proof. We then revise the argument and obtain an upper bound of $s_{\text{L}}(n) \leq \sqrt{n} + 5/4 + o(1)$ and establish that indeed, $s_{\text{L}}(n) = \sqrt{n} + 1$ for $n = k^2$, for $k \geq 2$, and so this formula holds for an infinite sequence of values of n.

The fact that the upper bound $s_{\text{L}}(n) \leq \sqrt{n} + 1 + o(1)$ does not follow from the argument in [2] was also noticed by Brenner and Vygen [1] in their elaborate study of planar networks with respect to various criteria of comparison. They also deduced the upper bound $s_{\text{L}}(n) \leq \sqrt{n} + 3/2 + o(1)$ [1, Corollary, p. 130] without, however, delving into the gaps of the argument in [2]. After an in-depth discussion of the matter, we give a new approach and a further reduction in the constant additive term in our Theorem 1.

Chung and Graham [2] also conjectured that $s_{\text{L}}(n) \leq \sqrt{n} + 1$ for all $n \geq 2$. Here we reduce their conjecture to a number-theoretic conjecture regarding integer partitions that we propose. We then verify the new conjecture for small values of n, and for a *new* infinite sequence of values of n. In Section 2 we prove the following.

Theorem 1. (i) *For every $n \geq 2$ we have $s_{\text{L}}(n) \leq \sqrt{n} + 5/4 + o(1)$.*

(ii) *For every $n \leq 50$ we have $s_{\text{L}}(n) \leq \sqrt{n} + 1$.*

(iii) *We also have: $s_{\text{L}}(k^2) \leq s_{\text{L}}(k^2+1) \leq k+1$, $s_{\text{L}}(k^2+2) \leq k+1+(k+1)^{-1}$, $s_{\text{L}}(k^2+k) \leq k+\frac{1}{2}-\frac{1}{2k+1}$, $s_{\text{L}}(k^2+k+1) \leq k+1/2$, and $s_{\text{L}}(k^2+2k) \leq k+1-\frac{1}{2k}$, for every $k \geq 2$; so in particular, the bound $s_{\text{L}}(n) \leq \sqrt{n}+1$ also holds for such n.*

We think that our proof method that establishes parts (ii) and (iii) could work for every n, but so far we have not been able to formulate a general argument. A combinatorial formulation and a relevant conjecture are proposed in Section 4.

2 Methods of proof

Our proofs are constructive and bear some similarities to earlier proofs. The first method of proof is geometric; it yields part (i) of Theorem 1. The second method of proof is purely combinatorial; it yields parts (ii) and (iii) of Theorem 1. We next present these two methods.

Few's method. In Few's proof [5] (as presented in a simpler but equivalent way by Chung and Graham [2]), the square U is subdivided into s strips by $s-1$ equally spaced horizontal segments $\ell_1, \ldots, \ell_{s-1}$ of unit length. Including the lower and upper sides of U yields $s+1$ horizontal unit segments $\ell_0, \ell_1, \ldots, \ell_{s-1}, \ell_s$. Then each point is in some strip (if a point is on the horizontal segment shared by two adjacent strips, associate the point with either strip, arbitrarily). For each point in some strip, take a vertical segment through the point to join it with the two horizontal segments bounding the strip. Take also the left and right sides of U. Then the n points are connected by $s+1$ horizontal and 2 vertical segments of length 1, and n vertical segments of length $1/s$. The total length of these segments is

$$(s+1) + 2 + n/s = (s + n/s) + 3. \tag{1}$$

In particular, the vertical segment of length $1/s$ through each point in a strip is the union of two vertical segments joined at the point, one connecting the point to a horizontal segment with an odd index, and the other connecting the point to a horizontal segment with an even index. From these segments we can construct two disjoint Steiner trees for the n points:

1. The left side of U, the horizontal segments with even indices, and the vertical segments connecting each point to a horizontal segment with an even index.

2. The right side of U, the horizontal segments with odd indices, and the vertical segments connecting each point to a horizontal segment with an odd index.

One of the two Steiner trees has length at most $(s+n/s)/2 + 3/2$. Note that

$$s + n/s = 2\sqrt{n} + O(1/\sqrt{n}), \quad \text{for } s = \sqrt{n} \pm O(1). \tag{2}$$

Thus by setting s to an integer near $\sqrt{n}$, e.g., $\lfloor \sqrt{n} \rfloor$ or $\lfloor \sqrt{n} \rfloor + 1$, the following bound is obtained:

$$s_{\text{L}}(n) \leq \sqrt{n} + 3/2 + o(1). \tag{3}$$

A more careful analysis [5, Theorem 2] yields a worst-case bound for all $n \geq 2$:

$$s_{\text{L}}(n) \leq \sqrt{n} + 7/4. \tag{4}$$

The attempt by Chung and Graham. Following the general approach due to Few [5], Chung and Graham [2] tried to improve the previous bound by omitting the left and right sides of U from the set of segments. To maintain connectivity of the two trees, they shift the horizontal segments, and extend/shrink the vertical segments through the points. We now review the transformation in [2]:

1. Translate each of the $s+1$ horizontal segments vertically, if necessary, until it is incident to at least one of the n points. When a horizontal segment is moved up or down, the vertical segments connecting it to the points in adjacent strips are extended or shrunk accordingly. The direction of the translation is chosen such that the total length of the segments do not increase.

2. Now each horizontal segment ℓ_i is incident to some point p_i whose vertical segment connects ℓ_i to either ℓ_{i-1} or ℓ_{i+1}. For $1 \le i \le s-1$, extend the vertical segment through p_i across ℓ_i until it connects ℓ_{i-1} and ℓ_{i+1}. Then all horizontal segments with odd indices are connected by the extended vertical segments across the horizontal segments with even indices, and all horizontal segments with even indices are connected by the extended vertical segments across the horizontal segments with odd indices. Then two disjoint Steiner trees can still be constructed from these segments, without the left and right side of U.

Chung and Graham [2] then claim that the increase in the total length of the segments in the second step of the transformation is at most 1. Combining this increase of 1 with the decrease of 2 due to the omission of the left and right sides of U, the total length of the segments becomes at most $(s+1)+(n/s+1)=(s+n/s)+2$, which leads to a bound of $s_{\llcorner}(n) \le \sqrt{n}+1+o(1)$. Unfortunately, this claim is false.

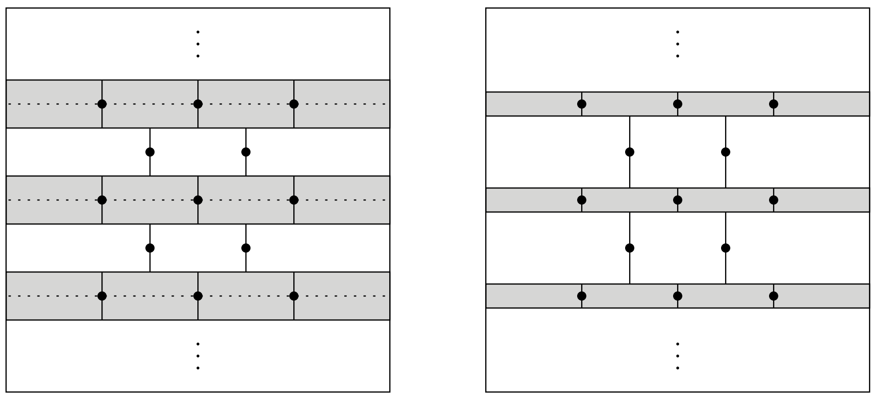

Figure 1: Transformation of alternating heavy and light strips.

Refer to Fig. 1. Before we apply the transformation, suppose that the s strips are alternately *heavy* and *light*: each heavy strip contains exactly $t+1$ points and each light strip contains exactly t points, for some $t \ge 3$. Moreover, suppose that the points in each strip (t and $t+1$, respectively) are placed vertically near the middle horizontal line of the strip. Then the first step of the transformation would translate the two horizontal segments bounding each heavy strip towards its middle line, shrinking the heavy strip to almost zero height, and expanding the adjacent light strips. Correspondingly, the $t+1$ vertical segments in each heavy strip become shorter, and the t vertical segments in each light strip become longer. The translation of each horizontal segment (except the top or the bottom one) causes a decrease in total length of the $2t+1$ adjacent vertical segments exactly equal to the distance of the translation. Thus the decrease in total length of all vertical segments due to the translations of all horizontal segments is roughly equal to the decrease in total height of the heavy strips, which is about 1/2. Then the second step of the transformation would extend vertical segments in heavy strips to adjacent light strips, resulting in an increase in total length roughly equal to two times the total height of the now expanded light strips, which is about 2. The net increase in total length of the segments is thus about $2-1/2=3/2>1$.

A concrete example is included for convenience: Let $n=k^2+(3/2)k$ for an even $k \ge 2$. Then $\sqrt{n}=k+3/4-o(1)$, and so $k+1$ is the integer closest to $\sqrt{n}$. Then by the choice before (19) in [2], the n points are partitioned into $k+1$ strips. By the construction of our counterexample, the n points are distributed in $k+1$ strips that are alternately heavy and light: $k/2$ heavy trips with $k+1$ points each; and $k/2+1$ light strips each with k points each. Note that $(k/2)(k+1)+(k/2+1)k=k^2+(3/2)k=n$.

So there are $k + 2$ horizontal segments, each of length 1; note that $k + 2$ is even. After the transformation, the total length of the vertical segments is at least $2 + k - o(1)$: indeed, $2 - o(1)$ accounts for the 2 extended vertical segments in each light strip, and $k - o(1)$ accounts for the k vertical segments through the k points in each light strip). Consequently, the total length of all segments after the transformation is at least $(k + 2) + (2 + k - o(1)) = 2(k + 2 - o(1))$, and each of the two trees has length at least

$$k + 2 - o(1) \geq \sqrt{n} + 5/4 - o(1).$$

Indeed, this inequality holds for each tree, since the number of horizontal segments in each tree is $(k + 2)/2$, and by construction (due to the placement of the points near the middle line of each strip) the total lengths of the vertical segments in the two trees differ by $o(1)$. Finally notice that all these properties could be replicated with respect to the other coordinate axis, if desired.

A revised approach. We next show that although the proof of Chung and Graham [2] is flawed, their idea of extending certain vertical segments to connect horizontal segments, instead of completely relying on the left and right side of U for connectivity, can still lead to an improvement over Few's bound. Consider the following two cases for any strip (except the top or the bottom one) in Few's construction.

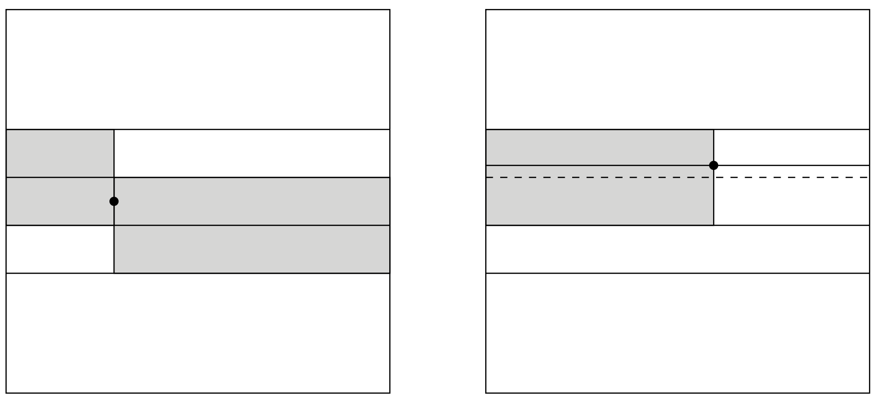

Figure 2: Left: Projections of two extended vertical segments through a point inside a non-empty strip. Right: Projection of an extended vertical segment through a point above an empty strip.

Refer to Fig. 2 (left). Suppose that this strip contains at least one vertical segment through some point. Then this vertical segment of length $1/s$ can be duplicated and then extended, with one copy extending into the strip above and the other copy extending into the strip below, resulting in two vertical segments each of length $2/s$. By omitting the original vertical segment of length $1/s$, and using these two extended vertical segments each of length $2/s$ to replace portions of the two vertical sides of U that correspond to their projections, respectively, two disjoint Steiner trees can still be constructed, while the total length of the segments decreases by $1/s$. More precisely, we use the extended vertical segment connecting two horizontal segments of *even* indices to replace its projection to the *left* side of U, and use the extended vertical segment connecting two horizontal segments of *odd* indices to replace its projection to the *right* side of U.

Refer to Fig. 2 (right). Suppose that this strip is empty but the adjacent strip from above is not empty. Then we can move the horizontal segment shared by the two strips up, as in the first step of the transformation in Chung and Graham [2], until it is incident to a point, and then extend the vertical segment through this point down, to length $2/s$. By omitting the original vertical segment of length $1/s$, and using the extended vertical segment of length $2/s$ to replace part of a vertical side of U that corresponds to its projection, two disjoint Steiner trees can still be constructed, while the total length of the segments decreases by $1/s$.

Note that if any two consecutive strips are empty, then the horizontal segment shared by the two strips has no vertical segments of length $1/s$ attached, and hence can be safely removed, thereby decreasing the total length of the segments to $s+2+n/s = (s+n/s)+2$ and leading to a bound of $s_{\llcorner}(n) \leq \sqrt{n}+1+o(1)$. Now suppose without loss of generality that no two consecutive strips are empty. Let us consider a subset of $s/2-O(1)$ non-consecutive strips (excluding the top and bottom strips) of the same parity, say, each of these strips is bounded by a horizontal segment of an even index from below. For each strip in this subset, apply the operations as in Fig. 2 (left) if it is not empty, and as in Fig. 2 (right) if it is empty. Since these strips are not consecutive, the projections of the extended vertical segments from these strips to the left and right sides of U, respectively, do not overlap. The decreases in the total length of the segments due to these operations then sum up to at least $(1/s)(s/2-O(1)) = 1/2 - O(1/s)$. The total length is thus at most

$$(s+1)+2+n/s-(1/2-O(1/s)) = s+n/s+5/2+O(1/s).$$

Recall (2); thus by setting s to an integer near $\sqrt{n}$, we obtain the following bound that improves (3):

$$s_{\llcorner}(n) \leq \sqrt{n}+5/4+o(1). \tag{5}$$

A new twist. Instead of subdividing U evenly by strip height, as done by Few, we subdivide the points evenly as possible among strips; we call such a subdivision an *even subdivision.* The number of strips s is either $\lfloor\sqrt{n}\rfloor$ or $\lfloor\sqrt{n}\rfloor+1$, and it remains unspecified until later in the proof. The strips may have unequal heights. One of the even subdivisions of this form is chosen in the end. Let $y_0 \leq y_1 \leq \cdots \leq y_{n-1}$ be the y-coordinates of the n points. Without loss of generality it can be assumed that $0 = y_0 \leq y_1 \leq \cdots \leq y_{n-1} = 1$ (otherwise, the bound only gets better). For $i = 1,\ldots,n-1$, let $x_i = y_i - y_{i-1}$, be the (length of the) ith *gap.* Obviously, we have

$$x_i \geq 0 \text{ for every } i = 1,\ldots,n-1, \text{ and } \sum_{i=1}^{n-1} x_i = 1. \tag{6}$$

Consider an even subdivision consisting of s strips. Let n_j-1 denote the number of points in strip j, $j = 1,\ldots,s$, where the $s+1$ points incident to the strip boundaries are not counted by the numbers n_j-1. We have $s+1+\sum_{j=1}^{s}(n_j-1) = n$, or equivalently

$$\sum_{j=1}^{s} n_j = n-1. \tag{7}$$

A subdivision can be described by the tuple $(n_1,\ldots,n_s)$ with the above sum condition; equivalently n_j is equal to the number of gaps covered by strip j, $j = 1,\ldots,s$. Observe that the gaps (their lengths and their number) are independent of the number s of strips in the subdivision we consider.

Since the subdivision is even, any two n_j differ by at most 1. Suppose that $t \leq n_j \leq t+1$, for $j = 1,\ldots,s$. More precisely, having chosen s, we set

$$t = t(s) = \left\lfloor \frac{n-1}{s} \right\rfloor. \tag{8}$$

The jth strip is called *heavy* if $n_j = t+1$ and *light* if $n_j = t$. Let z_j be the height of the jth strip, i.e., the sum of the gap-lengths in the jth strip, for $j = 1, 2, \ldots, s$. Clearly $\sum_{j=1}^{s} z_j = \sum_{i=1}^{n-1} x_i = 1$. Given an even subdivision Ξ, let $H(\Xi)$ denote the set of indices $1 \le i \le n-1$ for which the ith gap belongs to a heavy strip. Observe that

$$\sum_{j\colon n_j=t+1} z_j = \sum_{i\in H(\Xi)} x_i. \tag{9}$$

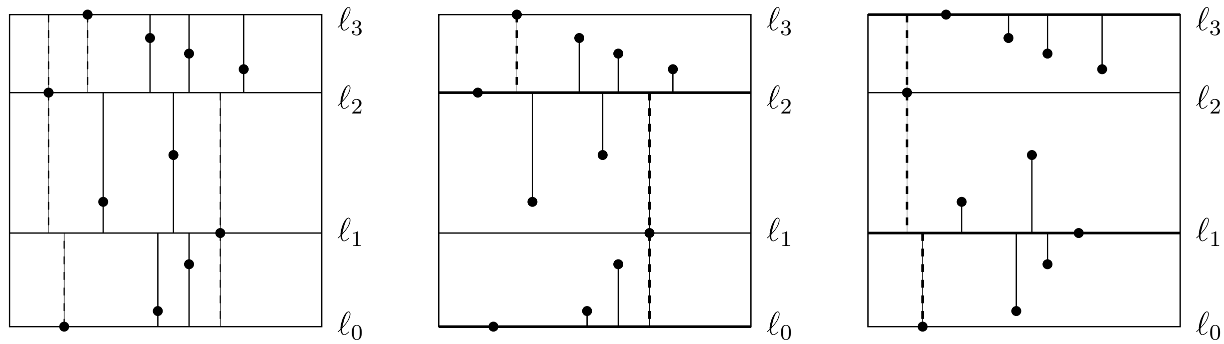

Figure 3: Left: an example of subdivision in the construction for $n = 11$ points; here we have $s = 3$, $t = 3$, and $(n_1, n_2, n_3) = (3, 3, 4)$. Middle and Right: the two rectilinear Steiner trees resulting from the construction.

Refer to Fig. 3. Put the horizontal segments ℓ_i, $0 \le i \le s$, into two sets, one with even indices, and the other with odd indices. Add vertical segments to each set to connect the points inside each strip to the bounding horizontal segment in that set. Doubly connect horizontal segments with consecutive indices by adding two (dashed) vertical segments incident to two points on the two segments, respectively, to the two sets. Then the two sets of segments form two rectilinear Steiner trees for the n points, and their total length is bounded from above by

$$\begin{aligned}(s+1) + \sum_{j=1}^{s}(n_j-1)z_j + 2 &= (s+1) - 1 + \sum_{j=1}^{s} t z_j + \sum_{j\colon n_j=t+1} z_j + 2 \\ &= s + t + \sum_{i\in H(\Xi)} x_i + 2. \end{aligned} \tag{10}$$

Hence the cost of one of the two trees does not exceed

$$\frac{s + t + \sum_{i\in H(\Xi)} x_i}{2} + 1. \tag{11}$$

To prove the conjectured inequality, it remains to show that one of the subdivisions is such that $\sum_{j\colon n_j=t+1} z_j$ is small; specifically, we will show that there is a subdivision Ξ with

$$L_0 := \frac{s + t + \sum_{i\in H(\Xi)} x_i}{2} \le \sqrt{n}. \tag{12}$$

and by (11) this will imply that

$$s_{\llcorner}(n) \le L_0 + 1 \le \sqrt{n} + 1, \tag{13}$$

as desired.

Toolbox. We first present two technical lemmas that are equivalent and dual to each other:

Lemma 1. *Let $\mathcal{S}$ be a (finite) multiset of subdivisions of U into horizontal strips (not necessarily having the same number of strips). If each of the $n-1$ gaps (the ith gap has length x_i, $i = 1, \ldots, n-1$) is covered by at most m heavy strips of subdivisions in $\mathcal{S}$, then there exists a subdivision $\Xi \in \mathcal{S}$ satisfying*

$$\sum_{i \in H(\Xi)} x_i \leq \frac{m}{|\mathcal{S}|}.$$

Proof. Put $\tau = m/|\mathcal{S}|$. By our assumption that each gap is covered by at most m heavy strips, and (6), we have

$$\sum_{\Xi \in \mathcal{S}} \sum_{i \in H(\Xi)} x_i \leq m \sum_{i=1}^{n-1} x_i = m.$$

Then the lemma follows by the pigeonhole principle. □

Lemma 2. *Let $\mathcal{S}$ be a (finite) multiset of subdivisions of U into horizontal strips (not necessarily having the same number of strips). If each of the $n-1$ gaps (the ith gap has length x_i, $i = 1, \ldots, n-1$) is covered by at least m light strips of subdivisions in $\mathcal{S}$, then there exists a subdivision $\Xi \in \mathcal{S}$ satisfying*

$$\sum_{i \in H(\Xi)} x_i \leq 1 - \frac{m}{|\mathcal{S}|}.$$

Proof. Put $\tau = 1 - m/|\mathcal{S}|$. Since every gap is covered by at least m light strips of subdivisions in $\mathcal{S}$, it follows that every gap is covered by at most $|\mathcal{S}| - m$ heavy strips of subdivisions in $\mathcal{S}$, and so the claimed inequality follows from Lemma 1. □

Remark. Similarly, Lemma 1 can be deduced from Lemma 2, and so the two lemmas are equivalent.

Observe that if $n = k^2$ and we set $s = k+1$, then $t = t(s) = k-1$, and thus $s + t = 2k$. The next lemma examines the value of $s + t$ over the remaining range of n:

Lemma 3. *Let $n = k^2 + r$, where $1 \leq r \leq 2k$. If $s = k$ or $s = k+1$, then*

$$s + t = \begin{cases} 2k & \text{if } r \leq k, \\ 2k+1 & \text{if } r \geq k+1. \end{cases}$$

Proof. Assume first that $s = k$. Then by (8),

$$t = \left\lfloor \frac{k^2 + r - 1}{k} \right\rfloor = k + \left\lfloor \frac{r-1}{k} \right\rfloor = \begin{cases} k & \text{if } r \leq k, \\ k+1 & \text{if } r \geq k+1. \end{cases}$$

Assume now that $s = k+1$. Then by (8),

$$t = \left\lfloor \frac{k^2 + r - 1}{k+1} \right\rfloor = k - 1 + \left\lfloor \frac{r}{k+1} \right\rfloor = \begin{cases} k-1 & \text{if } r \leq k, \\ k & \text{if } r \geq k+1. \end{cases}$$

In each case the lemma follows by adding the expressions of s and t. □

Remarks. 1. By Lemma 3, the sum $s+t$ in the numerators of the fractions in (12) is the same, provided that the subdivision used for $n = k^2 + r$, where $1 \le r \le 2k$ is one with either $s = k$ or $s = k+1$.

2. From Lemma 3, it is easy to show that $(s+t)/2 \le \sqrt{n}$. From (11), the cost of one of the two trees is at most $(s+t+3)/2$, since $\sum_{i \in H(\Xi)} x_i$ is at most 1. Therefore we have $(s+t+3)/2 = (s+t)/2 + 3/2 \le \sqrt{n} + 3/2$. This yields the following bound that improves (3) and (4), and complements (5):

$$s_{\text{L}}(n) \le \sqrt{n} + 3/2. \tag{14}$$

3 Case proofs

Let $n \ge 2$. We first consider some general cases.

$n = k^2$, for $k \ge 2$. We set $s = k+1$ and get $t = k-1$ and $n_1 = \cdots = n_{k+1} = t = k-1$ (a perfectly balanced subdivision). Consequently, by (12), we obtain $L_0 \le \frac{2k}{2} = k = \sqrt{n}$, as required.

$n = k^2 + 1$, for $k \ge 1$. We set $s = k$ and get $t = k$ and $n_1 = \cdots = n_k = k$ (a perfectly balanced subdivision). Consequently, by (12), we obtain $L_0 \le \frac{2k}{2} = k \le \sqrt{n}$, as required.

$n = k^2 + 2$, for $k \ge 2$. Consider the set $\mathcal{T}_1$ of subdivisions with $s = k$. By Lemma 3, we have $t = k$ and $(n_1, \ldots, n_k) = (k, \ldots, k, k+1)$, or a permutation of this tuple, i.e., one n_i is $k+1$ and the rest are k; we have $|\mathcal{T}_1| = \binom{k}{1} = k$. Consider also the set $\mathcal{T}_2$ of subdivisions with $s = k+1$ and get $t = k-1$, and $(n_1, \ldots, n_{k+1}) = (k-1, \ldots, k-1, k, k)$, or a permutation of this tuple, i.e., two n_i are k and the rest are $k-1$; we have $|\mathcal{T}_1| = \binom{k+1}{2}$. By taking all k subdivisions from $\mathcal{T}_1$ and one subdivision from $\mathcal{T}_2$, namely $(k, k-1, \ldots, k)$, each of the $n-1$ gaps is covered at most twice by heavy strips. By Lemma 1 with $|\mathcal{S}| = k+1$ and $m = 2$, we can set $\tau = \frac{m}{|\mathcal{S}|} = \frac{2}{k+1}$ and (12) yields

$$L_0 \le \frac{s+t+\tau}{2} = \frac{2k + \frac{2}{k+1}}{2} = k + \frac{1}{k+1} \le \sqrt{k^2+2},$$

as required.

$n = k^2 + k$, for $k \ge 2$. Consider the set $\mathcal{T}_1$ of subdivisions with $s = k$. By Lemma 3, we have $t = k$ and $(n_1, \ldots, n_k) = (k, k+1, \ldots, k+1)$, or a permutation of this tuple, i.e., one n_i is k and the rest are $k+1$; we have $|\mathcal{T}_1| = \binom{k}{1} = k$. Consider also the set $\mathcal{T}_2$ of subdivisions with $s = k+1$ and get $t = k-1$, and $(n_1, \ldots, n_{k+1}) = (k-1, k, \ldots, k)$, or a permutation of this tuple, i.e., one n_i is $k-1$ and the rest are k; we have $|\mathcal{T}_2| = \binom{k+1}{1} = k+1$. Let $\mathcal{S} = \mathcal{T}_1 \cup \mathcal{T}_2$. We claim that each integer $1 \le i \le k^2 + k - 1$ is covered by at least one light strip of a subdivision in $\mathcal{S}$. Indeed: write $i = kp + q$, where $0 \le p \le k$ and $0 \le q \le k-1$. If $q = 0$, we have $p \ge 1$; use a subdivision in $\mathcal{T}_1$ with $p-1$ heavy strips followed by a light strip covering i. If $q \ge 1$, use a subdivision in $\mathcal{T}_2$ with p heavy strips followed by a light strip covering i. By Lemma 2 with $|\mathcal{S}| = 2k+1$ and $m = 1$, we can set $\tau = 1 - \frac{m}{|\mathcal{S}|} = 1 - \frac{1}{2k+1}$ and get

$$L_0 \le \frac{k+k+\tau}{2} = k + \frac{1}{2} - \frac{1}{2k+1} \le \sqrt{k^2+k},$$

as required.

$n = k^2 + k + 1$, for $k \ge 1$. We set $s = k$ and get $t = k+1$ and $n_1 = \cdots = n_k = k+1$ (a perfectly balanced subdivision). Consequently, by (12), we obtain $L_0 \le \frac{2k+1}{2} = k + 1/2 \le \sqrt{n}$, as required.

$n = k^2 + 2k$, for $k \geq 1$. Consider the set $\mathcal{T}_2$ of subdivisions with $s = k+1$ and get $t = k$, and $(n_1, \ldots, n_{k+1}) = (k, k, k+1, \ldots, k+1)$, or a permutation of this tuple, i.e., two n_i are k and the rest are $k+1$; we have $|\mathcal{T}_2| = \binom{k+1}{2}$. Select in $\mathcal{S}$ the set of k subdivisions from $\mathcal{T}_2$ in which the two light parts are consecutive; now each of the $n-1$ gaps is covered at least once by light strips. By Lemma 2 with $|\mathcal{S}| = k$ and $m = 1$, we can set $\tau = 1 - \frac{m}{|\mathcal{S}|} = \frac{k-1}{k}$ and (12) yields

$$L_0 \leq \frac{s+t+\tau}{2} = \frac{(k+1)+k+\frac{k-1}{k}}{2} = k+1-\frac{1}{2k} \leq \sqrt{k^2+2k},$$

as required.

The proofs for the remaining uncovered cases (up to 50) are summarized in Table 1. For row n, one verifies that $L_0 \leq L_0^+ \leq \sqrt{n}$, where L_0^+ is the upper bound on L_0 resulting from (12) via Lemma 1 or Lemma 2.

n	$\mathcal{S}$	Lemma	$\lvert\mathcal{S}\rvert$	m	τ	L_0^+
14	$(4,5,4)$, $(4,3,3,3)$, $(3,3,3,4)$	Lem. 1	3	1	1/3	11/3
19	$(4,4,5,5)$, $(5,5,4,4)$, $(4,4,3,3,4)$	Lem. 2	3	1	2/3	13/3
22	$(5,4,4,4,4)$, $(4,4,5,4,4)$, $(4,4,4,4,5)$	Lem. 1	3	1	1/3	14/3
23	$(5,5,4,4,4)$, $(4,4,4,5,5)$	Lem. 1	2	1	1/2	19/4
28	$(5,5,5,6,6)$, $(6,6,5,5,5)$	Lem. 2	2	1	1/2	21/4
29	$(5,5,6,6,6)$, $(6,6,6,5,5)$, $(5,5,4,4,5,5)$	Lem. 2	3	1	2/3	16/3
32	$(7,6,6,6,6)$, $(6,7,6,6,6)$, $(6,6,6,6,7)$, $(6,6,6,7,6)$, $(6,5,5,5,5,5)$, $(5,5,5,5,5,6)$, $(5,5,6,5,5,5)$	Lem. 1	7	2	2/7	79/14
33	$(6,7,6,7,6)$, $(6,7,7,6,6)$, $(6,6,7,7,6)$, $(5,5,5,5,6,6)$, $(6,6,5,5,5,5)$, $(6,5,5,5,5,6)$, $(6,5,5,5,5,6)$	Lem. 1	7	3	3/7	40/7
34	$(6,7,7,7,6)$, $(5,5,5,6,6,6)$, $(6,6,6,5,5,5)$, $(6,5,5,5,6,6)$, $(6,6,5,5,5,6)$	Lem. 1	5	3	3/5	29/5
39	$(6,6,6,6,7,7)$, $(7,7,6,6,6,6)$, $(6,6,6,7,7,6)$, $(6,7,7,6,6,6)$, $(6,5,5,6,5,5,6)$, $(5,5,5,5,6,6,6)$, $(6,6,6,5,5,5,5)$	Lem. 1	7	3	3/7	87/14
40	$(6,6,6,7,7,7)$, $(7,7,7,6,6,6)$, $(6,6,7,6,7,7)$, $(7,7,6,7,6,6)$, $(5,5,5,6,6,6,6)$, $(6,6,6,6,5,5,5)$, $(5,6,5,5,6,6,6)$, $(6,6,6,5,5,6,5)$	Lem. 1	8	5	5/8	101/16
41	$(6,7,7,7,7,6)$, $(7,7,6,6,7,7)$, $(6,5,5,6,6,6,6)$, $(6,6,6,6,5,5,6)$	Lem. 2	4	1	3/4	51/8
44	$(7,6,6,6,6,6,6)$, $(6,6,7,6,6,6,6)$, $(6,6,6,6,7,6,6)$, $(6,6,6,6,6,6,7)$	Lem. 1	4	1	1/4	53/8
45	$(7,7,8,8,7,7)$, $(6,6,6,6,6,7,7)$, $(7,7,6,6,6,6,6)$	Lem. 1	3	1	1/3	20/3
46	$(6,6,6,6,7,7,7)$, $(7,7,7,6,6,6,6)$	Lem. 2	2	1	1/2	27/4
47	$(6,6,6,7,7,7,7)$, $(7,7,6,6,6,7,7)$, $(7,7,7,7,6,6,6)$	Lem. 2	3	1	2/3	41/6

Table 1: The remaining cases (up to 50).

4 A combinatorial formulation

It is useful to reinterpret our findings in combinatorial terms. Recall [14, Ch. 15] that an *ordered partition* (or *composition*) of a positive integer n is an integral solution, $(n_1, \ldots, n_k)$, of the following equation:

$$\sum_{i=1}^{k} n_i = n, \text{ where } n_i \geq 1, \text{ for } i = 1, \ldots, k. \tag{15}$$

If any two n_i differ by at most 1, we call such a solution an *even ordered partition.* If all n_i are the same, we say that we have a *perfect ordered partition.* Given an ordered partition as in (15), assign each integer i between 1 and n to one of the k parts, in increasing order:

$$\text{if } \sum_{h=1}^{j-1} n_h + 1 \leq i \leq \sum_{h=1}^{j} n_h \text{ for some } j, \quad 1 \leq j \leq k,$$

i is assigned to the jth part; we also say that i is covered by the jth part. For illustration, the ordered partition $(2,3,1,5)$ of 11 would generate the following partition of the integers in $\{1,2,\ldots,11\}$: $\{1,2\}$, $\{3,4,5\}$, $\{6\}$, $\{7,8,9,10,11\}$.

Consider now an even ordered partition where some n_i equal t and others equal $t+1$; we then call the parts equal to t *light* and those equal to $t+1$ *heavy.* We propose the following conjecture on ordered partitions of positive integers, which is clearly of independent interest.

Conjecture 1. *Let $n \geq 2$ be any natural number and let $s_1 = k = \lfloor \sqrt{n} \rfloor$, $s_2 = k+1$, and correspondingly set $t_j = t(s_j)$, $j = 1,2$, according to the formula $t = t(s) = \left\lfloor \frac{n-1}{s} \right\rfloor$ as in* (8)*. Write $\sigma = s_1 + t_1 = s_2 + t_2$ (by Lemma 3). Then there exists a multiset $\mathcal{S}$ of even ordered partitions of $n-1$ with light parts of size t_1 and/or t_2, with the following properties:*

(i) *Each integer in $\{1,2,\ldots,n-1\}$ is covered by at least m light parts of partitions in $\mathcal{S}$.*

(ii) *$\frac{\sigma+\tau}{2} \leq \sqrt{n}$, where $\tau = 1 - \frac{m}{|\mathcal{S}|}$.*

Remarks. 1. As explained in the second part of Section 2, Conjecture 1 would imply the old conjecture on the minimum length of rectilinear Steiner trees due to Chung and Graham, namely that $s_{\text{L}}(n) \leq \sqrt{n} + 1$.

2. As shown in Section 3, Conjecture 1 is verified for the sequence of values of n covered by Theorem 1.

3. Finally, observe that $\sigma, \tau \in \mathbb{Q}$ and so the left-hand-side expression in (ii) would give a rational approximation from below of $\sqrt{n}$.

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