A problem on track runners

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Abstract

Consider the unit circle C and a circular arc A of length $\ell = |A| < 1$. It is shown that there exists $k = k(\ell) \in \mathbb{N}$, and a schedule for k runners with k distinct but constant speeds so that at any time $t \geq 0$, at least one of the k runners is *not* in A.

Keywords: Kronecker's theorem, rational independence, track runners, multi-agent patrolling, idle time.

1 Introduction

In the classic *lonely runners conjecture*, introduced by Wills [11] and Cusick [4], k agents run clockwise along a circle of length 1, starting from the same point at time $t = 0$. They have distinct but constant speeds. A runner is called *lonely* when he/she is at distance of at least $\frac{1}{k}$ from any other runner (along the circle). The conjecture asserts that each runner a_i is lonely at some time $t_i \in (0, \infty)$. The conjecture has only been confirmed for up to $k = 7$ runners [1, 2]. A recent survey [7] lists a few other related problems.

Recently, some problems with similar flavor have appeared in the context of *multi-agent patrolling*, particularly in some one-dimensional scenarios [3, 5, 6, 9, 10]. Suppose that k mobile agents with (possibly distinct) maximum speeds v_i $(i = 1, \ldots, k)$ are in charge of *patrolling* a closed or open fence (modeled by a circle or a line segment). The movement of the agents over the time interval $[0, \infty)$ is described by a *patrolling schedule* (or *guarding schedule*), where the speed of the ith agent, $(i = 1, \ldots, k)$, may vary between zero and its maximum value v_i in any of the two directions along the fence. Given a closed or open fence of length ℓ and maximum speeds $v_1, \ldots, v_k > 0$ of k agents, the goal is to find a patrolling schedule that minimizes the *idle time*, defined as the longest time interval in $[0, \infty)$ during which some point along the fence remains unvisited, taken over all points. Several basic problems are open, such as the following: It is *not* known how to decide, given $v_1, \ldots, v_k > 0$, and $\ell, \tau > 0$ whether k agents with these maximum speeds can ensure an idle time at most τ when patrolling a segment of length ℓ.

This note is devoted to a question on track runners. In the spirit of the lonely runner conjecture, we posed the following question in [7]:

> Assume that k runners $1, 2, \ldots, k$, with distinct but constant speeds, run clockwise along a circle of length 1, starting from arbitrary points. Assume also that a certain half of the circular track (or any other fixed circular arc) is in the shade at all times. Does there exist a time when all runners are in the shade along the track?

Here we answer the question in the negative: the statement does not hold even if the shaded arc almost covers the entire track, e.g., has length 0.999, provided k is large enough.

Notation and terminology. We parameterize a circle of length ℓ by the interval $[0, \ell]$, where the endpoints of the interval $[0, \ell]$ are identified. A *unit* circle is a circle of unit length $C = [0, 1]$ mod 1. A *schedule* of k agents consists of k functions $f_i : [0, \infty] \to [0, \ell]$, for $i = 1, \ldots, k$, where $f_i(t)$ mod ℓ is the position of agent i at time t. Each function f_i is continuous, piecewise differentiable, and its derivative (speed) is bounded by $|f_i'| \leq v_i$. A schedule is called *periodic* with period $T > 0$ if $f_i(t) = f_i(t + T) \mod \ell$ for all $i = 1, \ldots, k$ and $t \geq 0$. $H_n = \sum_{i=1}^{n} 1/i$ denotes the nth *harmonic number*; and $H_0 = 0$.

2 Track runners in the shade

We first show that the general answer to the problem posed in [7] is negative:

Theorem 1. *Consider the unit circle C and a circular arc $A \subset C$ of length $\ell = |A| < 1$. Then there exists $k = k(\ell) \in \mathbb{N}$, and a schedule for k runners with k distinct constant speeds, so that at any time $t \geq 0$, at least one of the k runners is in the complement $C \setminus A$.*

Proof. Set $v_i = i$ as the speed of agent i, for $i = 1, \ldots, k$, where $k = k(\ell) \in \mathbb{N}$ is to be specified later. Assume, as we may, that $C \setminus A = [0, a]$, for some $a \in (0, 1)$. Let $t_0 = 0$. Since the speed of each agent is an integer multiple of the circle length $\text{len}(C) = 1$, the resulting schedule is periodic and the period is 1. To ensure that at any $t \geq 0$, at least one agent is in $[0, a]$, it suffices to ensure this *covering condition* on the time interval $[0, 1]$, i.e., one period of the schedule. All agents start at time $t = 0$; however, it is convenient to specify their schedule with their positions at later time.

Agent 1 starts at point 0 at time 0; at time a, its position is at a (exiting $[0, a]$). Agent 2 starts at point 0 at time a; at time $a + a/2$, its position is at a (exiting $[0, a]$). Agent 3 starts at point 0 at time $a + a/2$; at time $a + a/2 + a/3$, its position is at a (exiting $[0, a]$). Subsequent agents are scheduled according to this pattern. For $i = 1, \ldots, k$, agent i starts at point 0 at time aH_{i-1}; at time aH_i, its position is at a (exiting $[0, a]$). The schedules are given by the functions $f_i(t) = it - iaH_{i-1}$ for $i = 1, \ldots, k$.

The construction ensures that

1. agent i is in $[0, a]$ during the time interval $[t_{i-1}, t_i]$, for $i = 1, \ldots, k$.
2. $\bigcup_{i=1}^{k} [t_{i-1}, t_i] \supseteq [0, 1]$.

Indeed, condition 2 is $aH_k \geq 1$, or equivalently $H_k \geq 1/a$. Since $\ln k \leq H_k$, it suffices to have $\ln k \geq 1/a$, or $k \geq \exp(1/a)$, and the theorem is proved. □

Now that we have seen that the general answer is negative, it is however interesting to exhibit some scenarios when the result holds.

A set of numbers $\xi_1, \xi_2, \ldots, \xi_k$ are said to be *rationally independent* if no linear relation

$$a_1\xi_1 + a_2\xi_2 + \cdots + a_k\xi_k = 0,$$

with integer coefficients, not all of which are zero, holds. In particular, if $\xi_1, \xi_2, \ldots, \xi_k$ are rationally independent, then they are pairwise distinct. Recall now Kronecker's theorem; see, e.g., [8, Theorem 444, p. 382].

Theorem 2. (Kronecker, 1884) *If $\xi_1, \xi_2, \dots, \xi_k \in \mathbb{R}$ are rationally independent, $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ are arbitrary, and T and ε are positive reals, then there is a real number $t > T$, and integers $p_1, p_2, \dots, p_k$, such that*

$$|t\xi_m - p_m - \alpha_m| \leq \varepsilon \quad (m = 1, 2, \dots, k).$$

As a corollary, we obtain the following result.

Theorem 3. *Assume that k runners $1, 2, \dots, k$, with constant rationally independent (thus distinct) speeds $\xi_1, \xi_2, \dots, \xi_k$, run clockwise along a circle of length 1, starting from arbitrary points. For every circular arc $A \subset C$ and for every $T > 0$, there exists $t > T$ such that all runners are in A at time t.*

Proof. Assume, as we may, that $A = [0, a]$, for some $a \in (0, 1)$. Let $0 \leq \beta_i < 1$, be the start position of runner i, for $i = 1, 2, \dots, k$. Set $\alpha_i = a/2 + 1 - \beta_i$, for $i = 1, 2, \dots, k$, set $\varepsilon = a/3$, and employ Theorem 2 to finish the proof. □

Remark. It is interesting to note that Theorem 1 gives a negative answer regardless of how long the shaded arc is, while Theorem 3 gives a positive answer regardless of how short the shaded arc is *and* for how far in the future one desires.

Observe that if $\xi_1, \xi_2, \dots, \xi_k$ are rationally independent reals, then at least one ξ_i must be irrational (in fact, all but at most one ξ_i must be irrational). To obtain the conclusion of Theorem 3 neither the condition that the speeds $\xi_1, \xi_2, \dots, \xi_k$ are rationally independent, nor the condition that at least one ξ_i is irrational are necessary. For instance, a condition imposed on the relative speeds suffices as it is shown in the following.

Theorem 4. *Assume that k runners $1, 2, \dots, k$, with constant but distinct speeds run clockwise along a circle of length 1, starting from arbitrary points. For every circular arc $A \subset C$, there exist suitable distinct speeds $v_1, v_2, \dots, v_k > 0$, so that for every $T > 0$, there exists $t > T$ such that all runners are in A at time t.*

Proof. Assume, as we may, that $A = [0, a]$, for some $a \in (0, 1)$. Let $\beta_1, \beta_2, \dots, \beta_k$ be the starting points of the runners, where $0 \leq \beta_i < 1$, for $i = 1, 2, \dots, k$. We proceed by induction on the number of runners k, and with a stronger induction hypothesis extending to every arc A. The base case $k = 1$ is satisfied by setting $v_1 = 1$ for any interval. The subsequent speeds will be set to increasing values, so that $v_1 < v_2 < \cdots < v_k$.

For the induction step, assume that the statement holds for runners $1, 2, \dots, k - 1$, the arc $A' = [0, a/2]$ and T, and we need to prove it for runners $1, 2, \dots, k$, the arc $A = [0, a]$ and T. By the induction hypothesis, there exists $t > T$ so that runners $1, 2, \dots, k - 1$, are in A' at time t. Set $v_k = \frac{2}{a} v_{k-1}$. Observe that runner k will enter the arc A at point 0 before any of the first $k - 1$ runners exits A at point a, regardless of his or her starting point. Hence all k runners will be in A at some time in the interval $[t, t + 1/v_k]$, completing the induction step, and thereby the proof of the theorem. □

References

[1] J. Barajas and O. Serra, The lonely runner with seven runners, *Electron. J. Combin.* **15** (2008), R48.

[2] T. Bohman, R. Holzman, and D. Kleitman, Six lonely runners, *Electron. J. Combin.* **8** (2001), R3.

[3] Y. Chevaleyre, Theoretical analysis of the multi-agent patrolling problem, in *Proc. Int. Conf. Intelligent Agent Technology (IAT 2004)*, IEEE, 2004, pp. 302–308.

[4] T. W. Cusick, View-obstruction problems, *Aequationes Math.* **9** (1973), 165–170.

[5] J. Czyzowicz, L. Gasieniec, A. Kosowski, and E. Kranakis, Boundary patrolling by mobile agents with distinct maximal speeds, in *Proc. 19th European Sympos. Algorithms (ESA 2011)*, LNCS 6942, Springer, 2011, pp. 701–712.

[6] A. Dumitrescu, A. Ghosh, and C. D. Tóth, On fence patrolling by mobile agents, *Electron. J. Combin.* **21(3)** (2014), P3.4.

[7] A. Dumitrescu and Cs. D. Tóth, Computational Geometry Column 59, *SIGACT News Bulletin* **45(2)** (2014), 68–72.

[8] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, fifth edition, Oxford University Press, 1979.

[9] A. Kawamura and Y. Kobayashi, Fence patrolling by mobile agents with distinct speeds, *Distributed Computing* **28** (2015), 147–154; an earlier version in *Proc. 23rd Int. Sympos. Algorithms and Computation (ISAAC 2012)*, LNCS 7676, Springer, 2012, pp. 598–608.

[10] A. Kawamura and M. Soejima, Simple strategies versus optimal schedules in multi-agent patrolling, in *Proc. 9th International Conference on Algorithms and Complexity (CIAC 2015)*, LNCS 9079, Springer, 2015, pp. 261–273.

[11] J. M. Wills, Zwei Sätze über inhomogene diophantische Approximation von Irrationalzahlen, *Monatsch. Math.* **71** (1967), 263–269.