On distinct distances among points in general position and other related problems

Adrian Dumitrescu*

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Abstract

A set of points in the plane is said to be in *general position* if no three of them are collinear and no four of them are cocircular. If a point set determines only distinct vectors, it is called *parallelogram free*. We show that there exist n-element point sets in the plane in general position, and parallelogram free, that determine only $O(n^2/\sqrt{\log n})$ distinct distances. This answers a question of Erdős, Hickerson and Pach. We then revisit an old problem of Erdős : given any n points in the plane (or in d dimensions), how many of them can one select so that the distances which are determined are all distinct? — and provide (make explicit) some new bounds in one and two dimensions. Other related distance problems are also discussed.

Keywords: distinct distances, distinct vectors, general position, Sidon sequence

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1 Introduction

In 1946, in his classical paper [11] published in the *American Mathematical Monthly*, Erdős raised the following question: What is the minimum number of distinct distances determined by n points in the plane? Denoting this number by $g(n)$, he proved that $g(n) = \Omega(\sqrt{n})$, and showed that $g(n) = O(n/\sqrt{\log n})$ by estimating the number of distinct distances in a $\sqrt{n} \times \sqrt{n}$ piece of the integer grid. He also went further to conjecture that the upper bound is best possible, in other words $g(n) = \Omega(n/\sqrt{\log n})$. The problem galvanized the interest of many researchers, and one can say that progress on this problem lead to many discoveries in related areas over time. The lower bound estimates have been successively raised by Moser [31], Chung [7], Beck [4], Clarkson, Edelsbrunner, Guibas, Sharir and Welzl [9], Chung, Szemerédi and Trotter [8], Székely [39], Solymosi and C. Tóth [38], Tardos [40], Katz and Tardos [26], with the current best lower bound standing at $g(n) \approx \Omega(n^{0.8641})$. The above question has lead to many other variants, some of which we discuss here.

Throughout this paper, we say that a set S of points in the plane is in *general position* if no three of them are collinear and no four of them are on a circle.[1] In our bounds, we denote by c possibly different absolute constants.

Erdős asked in 1985 whether there exist n points in general position that determine only $o(n^2)$ distinct distances [14]. Erdős, Hickerson and Pach [20] constructed such point sets with $O(n^{\log 3/\log 2})$ distinct

*Department of Computer Science, University of Wisconsin-Milwaukee, WI 53201-0784, USA. E-mail: ad@cs.uwm.edu
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[1] It is not uncommon to find this qualification attached to sets satisfying only one of the two restrictions. Alternatively, one can use the term *strong general position* for sets satisfying both restrictions.

distances, a bound that was later improved by Erdős, Füredi, Pach and Ruzsa [18] to $n2^{c\sqrt{\log n}}$ (for some constant c); see also [33]. It is still an open question whether this number can be linear in n. These constructions use many duplicate vectors, which motivates the further restriction of *no parallelogram*—equivalently, that no two vectors determined by the point set are the same. Erdős, Hickerson and Pach [20] raised the following question: Does there exist a set S of n points in the plane in general position, such that S does not contain all four vertices of a parallelogram, but $g(S)$, the number of distinct distances determined by S, is $o(n^2)$? The question also appears in a paper [16] from 1988, as well as in the recent collection of open problems [6] (Problem 3, Section 5.5, pp. 215). Here we give a positive answer and thus show that the above (three) conditions are not enough to force a quadratic number of distinct distances. Other tentative conditions enforcing a quadratic number of distinct distances, are discussed in Section 4.

For a prime p, and $x \in \mathbb{Z}$, let $\widehat{x} := x \mod p$ (we view $\widehat{x}$ as an element of $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$). An old construction of Erdős described below (see also [3, pp. 28–29], and [6, pp. 417]) has proved to be instrumental in answering several different questions in combinatorial geometry. Let n be a prime and consider the n-element point set $E_n = \{(i, \widehat{i^2}) \mid i = 0, 1, \ldots, n-1\}$. E_n is a subset of $G_n = \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, n-1\}\}$. An old (still unsolved) question asks how many points can be selected from the $n \times n$ grid G_n so that no three are collinear. Erdős has shown that E_n has no three collinear points, so this gives a first large set with n points; more complicated constructions approach $2n$ from below (obviously $2n$ is an upper bound). The set E_n gives also a first partial answer in the old Heilbronn problem: What is the smallest $a(n)$ such that any set of n points in the unit square determines a triangle whose area is at most $a(n)$? Heilbronn's question is in other words to find a set of n points in the unit square that is as far as possible from containing three collinear points; see [6, pp. 443]. Using the fact that the minimum nonzero area of a triangle in G_n is $1/2$, after suitable scaling E_n to the unit square, one gets the estimate $a(n) \geq \frac{1}{2(n-1)^2}$, i.e., half of the area of a scaled lattice square; see [3, 6]. A more complicated construction due to Komlós, Pintz and Szemerédi [27] yields the current best lower bound $a(n) = \Omega(\frac{\log n}{n^2})$.

Here we use the Erdős construction yet one more time, and show that a suitably large subset $S_n \subset E_n$ is in general position and parallelogram-free ($|S_n| = (n-1)/4$). The fact that S_n determines only $o(n^2)$ distinct distances comes out easily from the old Erdős upper bound of $g(n) = O(n/\sqrt{\log n})$. Let $v(n) = \min g(S)$, where the minimum is taken over all n-element point sets in the plane in general position, and parallelogram free.

Theorem 1 *For every natural number n, $v(n) = O(n^2/\sqrt{\log n})$.*

In the second part of our paper, we discuss another old problem of Erdős on distinct distances. As proved by Erdős [12], one can select an infinite subset of points from any infinite set of points in the plane such that all pairwise distances determined by the subset are distinct. Erdős also asked [13, 19, 21]: What is the largest number $h(n)$ so that any set of n points in the plane (or in d dimensions) has an $h(n)$-element subset in which all $\binom{h(n)}{2}$ distances are distinct? Denote this number by $h_d(n)$, and also write $h(n)$ for $h_2(n)$.

The problem on the line ($d = 1$) turns out to be related to the classical *Sidon sequences* [36]. Erdős conjectured that $h_1(n) = (1 + o(1))n^{1/2}$ [21], and observed that the upper bound follows from his 1941 result with Turán [23] on Sidon sequences. By combining various number theoretical results (some of them quite old and possibly forgotten), in particular a powerful result of Komlós et al. [28], one can show:

Theorem 2 *Given a set S of n points in the line, one can select a subset $X \subseteq S$ of size $|X| = \Omega(n^{1/2})$ in which all pairwise distances are distinct. This bound is best possible apart from a constant factor. Thus $h_1(n) = \Theta(n^{1/2})$; more precisely: $(0.0805 + o(1)) \cdot n^{1/2} \leq h_1(n) \leq (1 + o(1)) \cdot n^{1/2}$.*

For the planar variant, a $\sqrt{n} \times \sqrt{n}$ section of the integer grid yields $h(n) = O(n^{1/2}(\log n)^{-1/4})$; see also [6]. From the results in [2], it follows that $h(n) = \Omega(n^{1/5})$. The lower bound has been subsequently raised by Lefmann and Thiele [29] to $h(n) = \Omega(n^{1/4})$. By using their method in combination with recent results of Pach and Tardos [34] on the maximum number of isosceles triangles determined by a planar point set, a better bound can be derived. We have included a short outline of the argument in Section 3. Letting $\alpha = \frac{234-68e}{110-32e}$, where e is the base of the natural logarithm ($\alpha < 2.136$), Pach and Tardos proved that the number of isosceles triangles determined by a planar set of n points is $O(n^{\alpha+\varepsilon}) = O(n^{2.136})$, for any $\varepsilon > 0$; see [6, 26]. Put now $\beta = 1 - \frac{\alpha}{3} > 0.288$. The improved lower bound on $h(n)$ is:

Theorem 3 *For any $\varepsilon > 0$, out of any set S of n points in the plane, one can select a subset $X \subseteq S$ of size $|X| = \Omega(n^{\beta-\varepsilon}) = \Omega(n^{0.288})$ in which all pairwise distances are distinct. Thus $h(n) = \Omega(n^{\beta-\varepsilon}) = \Omega(n^{0.288})$.*

For $d \geq 3$, the lower bound $h_d(n) = \Omega(n^{1/(3d-2)})$ is known, cf. [29, 41]; see also [6].

2 Proof of Theorem 1

Assume first that n is a prime. Let $S_n = \{(i, \widehat{i^2}) \mid i = 0, 1, \ldots, (n-1)/4\}$. Recall that a $\sqrt{n} \times \sqrt{n}$ piece of the integer lattice determines $O(n/\sqrt{\log n})$ distances (cf. Erdős, this leads to the upper bound $g(n) = O(n/\sqrt{\log n})$). Therefore G_n determines $O(n^2/\sqrt{\log n})$ distinct distances; obviously this upper bound also holds for any subset of G_n, E_n or S_n in particular. Clearly S_n has no three collinear points (as a subset of E_n). To conclude our result in Theorem 1 it remains to be shown that S_n has no four points cocircular and that it determines no parallelogram. Then the result follows from standard facts about the distribution of primes [25], e.g., that there is a prime between k and $2k$ for any integer $k \geq 1$.

Lemma 1 *S_n determines no parallelogram.*

Proof. Assume (for contradiction) that $ABCD$ is a parallelogram whose vertices are in S_n. Let $A = (a, \widehat{a^2})$, $B = (b, \widehat{b^2})$, $C = (c, \widehat{c^2})$, $D = (d, \widehat{d^2})$. We may assume that $0 \leq a < b < c < d \leq (n-1)/4$. Observe that AD must be a diagonal, henceforth BC is the other diagonal. Denote by $x(p)$ and $y(p)$ the x- and y-coordinates of a point p. Since the midpoints of the diagonals coincide, we have

$$\frac{x(A)+x(D)}{2} = \frac{x(B)+x(C)}{2}, \text{ and } \frac{y(A)+y(D)}{2} = \frac{y(B)+y(C)}{2}.$$

The first equality yields $a + d = b + c$, or equivalently $b - a = d - c$, and note that $b - a$ is a nonzero invertible element of $\mathbb{Z}_n$ (since n is prime). The second equality yields $\widehat{a^2} + \widehat{d^2} = \widehat{b^2} + \widehat{c^2}$, or equivalently $\widehat{b^2} - \widehat{a^2} = \widehat{d^2} - \widehat{c^2}$. Taking this equality modulo n we get

$$(b-a)(b+a) \equiv (d-c)(d+c) \pmod{n}.$$

After simplifying by $b - a$, we obtain $a + b = c + d$, obviously a contradiction, since $1 \leq a + b < 2b < 2c < c + d < (n-1)/2$. □

Lemma 2 *S_n has no four points cocircular.*

Proof. Assume (for contradiction) that A, B, C, and D are four cocircular points of S_n. As in Lemma 1, let $A = (a, \widehat{a^2})$, $B = (b, \widehat{b^2})$, $C = (c, \widehat{c^2})$, $D = (d, \widehat{d^2})$, where $0 \leq a < b < c < d \leq (n-1)/4$. Let ℓ_{ab}, ℓ_{bc} and ℓ_{cd} denote the three (distinct) lines determined by the segments AB, BC and CD. Observe that none of these three lines is horizontal: this follows from the assumption $0 \le a < b < c < d \le (n-1)/4$. Let ℓ'_{ab}, ℓ'_{bc} and ℓ'_{cd} denote the three perpendicular bisectors of ℓ_{ab}, ℓ_{bc} and ℓ_{cd} respectively. Let the equation of ℓ'_{ab} be $\ell'_{ab} : y = m_{ab}x + n_{ab}$. The slope of ℓ_{ab} is $(\widehat{b^2} - \widehat{a^2})/(b-a)$, hence the slope of the perpendicular bisector ℓ'_{ab} is $m_{ab} = -(b-a)/(\widehat{b^2} - \widehat{a^2})$; observe that the denominator of the fraction is nonzero. By imposing the condition that ℓ'_{ab} is incident to the midpoint of AB, we have

$$\frac{\widehat{a^2} + \widehat{b^2}}{2} = -\frac{(b-a)(b+a)}{2(\widehat{b^2} - \widehat{a^2})} + n_{ab}.$$

This yields

$$n_{ab} = \frac{(\widehat{b^2})^2 - (\widehat{a^2})^2 + (b^2 - a^2)}{2(\widehat{b^2} - \widehat{a^2})}.$$

Hence the equation of ℓ'_{ab} is:

$$\ell'_{ab} : \quad y = -\frac{b-a}{\widehat{b^2} - \widehat{a^2}}x + \frac{(\widehat{b^2})^2 - (\widehat{a^2})^2 + (b^2 - a^2)}{2(\widehat{b^2} - \widehat{a^2})}.$$

Similarly, we get the equations of ℓ'_{bc} and ℓ'_{cd}:

$$\ell'_{bc} : \quad y = -\frac{c-b}{\widehat{c^2} - \widehat{b^2}}x + \frac{(\widehat{c^2})^2 - (\widehat{b^2})^2 + (c^2 - b^2)}{2(\widehat{c^2} - \widehat{b^2})}.$$

$$\ell'_{cd} : \quad y = -\frac{d-c}{\widehat{d^2} - \widehat{c^2}}x + \frac{(\widehat{d^2})^2 - (\widehat{c^2})^2 + (d^2 - c^2)}{2(\widehat{d^2} - \widehat{c^2})}.$$

Observe that the set $\{\ell'_{ab}, \ell'_{bc}, \ell'_{cd}\}$ has at least two distinct elements. Moreover, observe that the four points A, B, C, and D are cocircular if and only if ℓ'_{ab}, ℓ'_{bc} and ℓ'_{cd} are concurrent (i.e., incident to the center of the circle). Consider now a standard duality transform D which maps a nonvertical line ℓ with equation $y = rx + s$ to the point $\ell^* = (r, -s)$. Then three (not necessarily distinct, but with at least two distinct elements) lines ℓ_1, ℓ_2, ℓ_3 are concurrent if and only if the dual points $\ell_1^*, \ell_2^*, \ell_3^*$ are collinear. Therefore the following determinant of order 3 vanishes:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \frac{b-a}{\widehat{b^2}-\widehat{a^2}} & \frac{c-b}{\widehat{c^2}-\widehat{b^2}} & \frac{d-c}{\widehat{d^2}-\widehat{c^2}} \\ \frac{(\widehat{b^2})^2-(\widehat{a^2})^2+(b^2-a^2)}{2(\widehat{b^2}-\widehat{a^2})} & \frac{(\widehat{c^2})^2-(\widehat{b^2})^2+(c^2-b^2)}{2(\widehat{c^2}-\widehat{b^2})} & \frac{(\widehat{d^2})^2-(\widehat{c^2})^2+(d^2-c^2)}{2(\widehat{d^2}-\widehat{c^2})} \end{vmatrix} = 0.$$

After multiplying this equation by $(\widehat{b^2} - \widehat{a^2})(\widehat{c^2} - \widehat{b^2})(\widehat{d^2} - \widehat{c^2})$ we have

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{vmatrix} = 0,$$

where

$$\begin{aligned}
D_{21} &= (b-a)(\widehat{c^2}-\widehat{b^2})(\widehat{d^2}-\widehat{c^2}),\\
D_{22} &= (c-b)(\widehat{b^2}-\widehat{a^2})(\widehat{d^2}-\widehat{c^2}),\\
D_{23} &= (d-c)(\widehat{b^2}-\widehat{a^2})(\widehat{c^2}-\widehat{b^2}),\\
D_{31} &= \left[(\widehat{b^2})^2-(\widehat{a^2})^2+b^2-a^2\right](\widehat{c^2}-\widehat{b^2})(\widehat{d^2}-\widehat{c^2}),\\
D_{32} &= \left[(\widehat{c^2})^2-(\widehat{b^2})^2+(c^2-b^2)\right](\widehat{b^2}-\widehat{a^2})(\widehat{d^2}-\widehat{c^2}),\\
D_{33} &= \left[(\widehat{d^2})^2-(\widehat{c^2})^2+(d^2-c^2)\right](\widehat{b^2}-\widehat{a^2})(\widehat{c^2}-\widehat{b^2}).
\end{aligned}$$

Expanding Δ by the first row gives

$$\Delta = (D_{22}D_{33}-D_{23}D_{32})-(D_{21}D_{33}-D_{23}D_{31})+(D_{21}D_{32}-D_{22}D_{31})=0.$$

It follows that Δ must be also zero modulo n. A straightforward (but lengthy calculation) yields

$$\begin{aligned}
&D_{22}D_{33}-D_{22}D_{31}\equiv\\
\equiv\ &(c-b)(b^2-a^2)(d^2-c^2)(c^2-b^2)\left[(d^4-c^4+d^2-c^2)(b^2-a^2)-(b^4-a^4+b^2-a^2)(d^2-c^2)\right]\\
\equiv\ &(c-b)(b^2-a^2)(d^2-c^2)(c^2-b^2)\left[(d^2-c^2)(b^2-a^2)(d^2+c^2-a^2-b^2)\right]\\
\equiv\ &(c-b)(b^2-a^2)^2(d^2-c^2)^2(c^2-b^2)(c^2+d^2-a^2-b^2) \pmod{n}.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
-D_{23}D_{32}+D_{23}D_{31} &\equiv -(d-c)(b^2-a^2)^2(c^2-b^2)^2(d^2-c^2)(c^2-a^2) \pmod{n}.\\
-D_{21}D_{33}+D_{21}D_{32} &\equiv -(b-a)(c^2-b^2)^2(d^2-c^2)^2(b^2-a^2)(d^2-b^2) \pmod{n}.
\end{aligned}$$

Putting all these together yields

$$\Delta \equiv -(b-a)^2(c-b)^2(d-c)^2(d-b)(d-a)(c-a)(a+b)(b+c)(c+d)(a+b+c+d)\equiv 0 \pmod{n}.$$

However this is impossible by our choice of a, b, c, d, and by the primality of n. □

Note. After we obtained our result we discovered that the property of S_n shown in Lemma 2 was first established by Thiele [42] (who gave a different proof). This was in the context of finding large subsets of the $n \times n$ grid without four cocircular points in response to a problem raised by Erdős and Purdy; see [6, pp. 418].

3 Further connections and related problems

A *Sidon sequence* of integers $1 \le a_1 < a_2 < \cdots < a_s \le n$ is one in which the sums of all pairs, $a_i + a_j$, for $i \le j$, are all different [22, 24, 35]. Suppose that we were to find a large subset $A \subseteq \{1, 2, \ldots, n\}$ in which all distances are distinct. It is easy to see that this amounts to finding a large Sidon sequence in $\{1, 2, \ldots, n\}$: that is, A is a Sidon sequence if and only if the differences (distances) between any two elements are distinct. Indeed, if $a_i < a_j \le a_k < a_l$, then $a_j - a_i = a_l - a_k$ if and only if $a_i + a_l = a_j + a_k$ (the case of overlapping intervals is similar).

Denote by $s = s(n)$ the maximum number of elements in a Sidon sequence with elements not greater than n. By a packing argument, it follows that $s < 2n^{1/2}$, see [22], and this implies the same upper bound on $h_1(n)$: therefore $h_1(n) = O(n^{1/2})$ (for the moment, we do not insist on the constant). Sharper bounds (with a better constant) have been obtained by Erdős and Turán [23] and Lindström [30]: $s(n) \leq n^{1/2} + n^{1/4} + 1$. From the other direction, the existence of perfect difference sets [37] shows that $s(n) \geq (1-\varepsilon)n^{1/2}$ [19]. The reader can find more details in [22]. Let $S : 1 \leq a_1 < a_2 < \cdots < a_n$ be any sequence of n integers. By extending the previous lower bound (in a very broad setting), Komlós et al. [28] have proved that S always contains a Sidon subsequence of size $\Omega(n^{1/2})$. From their very general theorem, the resulting constant factor is quite small, about 2^{-15}, but this has been later raised by Abbott [1] to about $0.0805 \gtrapprox 2/25$. Therefore, out of a set of n integer points, one can always find a subset of size $\Omega(n^{1/2})$, with all distinct distances, and this is best possible.

3.1 All distinct distances on the line: proof of Theorem 2

It only remains to show that given n points on the line, one can select a subset of size $\Omega(n^{1/2})$, with all distinct distances. Let $A = \{a_1 < \ldots < a_n\}$ be a set of n points on the line. Using simultaneous approximation [25] (see other applications in [10, 32]), construct a set of n rational points $A' = \{a'_1 < \ldots < a'_n\}$, and then a set of integer points $A'' = \{a''_1 < \ldots < a''_n\}$, so that $a''_j - a''_i = a''_l - a''_k$ holds whenever $a_j - a_i = a_l - a_k$ holds. For any positive integer m, there exist n rational points $A' = \{a'_1 < \ldots < a'_n\} = \{r_1/m, \ldots, r_n/m\}$, where $r_i, m \in \mathbb{N}$, and

$$\left| a_i - \frac{r_i}{m} \right| \leq \frac{1}{m^{1+1/n}}, \quad 1 \leq i \leq n.$$

One can also ensure that the order of the points in A' is $r_1/m < \ldots < r_n/m$ (i.e., $a'_i = r_i/m$, for all i). We show there exists an m large enough such that the structure of distinct distances is preserved, and in particular we have:

$$g(A'') = g(A') = g(A). \tag{1}$$

Assume first that a pair of equal distances in A yields a pair of distinct distances for the corresponding points in A': that is, for $i < j \leq k < l$, $|a_j - a_i| = |a_l - a_k|$ (since any pair of equal distances yields a pair whose corresponding intervals are non-overlapping in their interiors), and without loss of generality, say,

$$\frac{r_l - r_k}{m} > \frac{r_j - r_i}{m} > 0. \tag{2}$$

We have

$$\frac{r_l - r_k}{m} = \left| \frac{r_l}{m} - a_l + a_l - \frac{r_k}{m} + a_k - a_k \right| \leq |a_l - a_k| + \frac{2}{m^{1+1/n}}, \tag{3}$$

and

$$\frac{r_j - r_i}{m} = \left| \frac{r_j}{m} - a_j + a_j - \frac{r_i}{m} + a_i - a_i \right| \geq |a_j - a_i| - \frac{2}{m^{1+1/n}}. \tag{4}$$

The above two inequalities imply

$$0 < (r_l - r_k) - (r_j - r_i) \leq m|a_l - a_k| + \frac{2}{m^{1/n}} - m|a_j - a_i| + \frac{2}{m^{1/n}} = \frac{4}{m^{1/n}}. \tag{5}$$

As m tends to infinity, this leads to $(r_l - r_k) = (r_j - r_i)$, a contradiction. A similar argument shows that there exists m such that also $g(A') \geq g(A)$ holds, and hence $g(A') = g(A)$. Through multiplication of all the numbers in A' by m, we obtain a set of n integers A'', such that $g(A') = g(A'')$. Thus (1) holds for m large enough.

Now select a large Sidon subsequence in A'' (cf. with the above result of [28], of size $\Omega(n^{1/2})$), and construct a subset $B \subset A$ by including all corresponding points from A into B. By the properties of A'', all pairwise distances in B are distinct, as desired. Taking now the best constants available into account, one has $(0.0805 + o(1)) \cdot n^{1/2} \leq h_1(n) \leq (1 + o(1)) \cdot n^{1/2}$.

3.2 All distinct distances in the plane: proof of Theorem 3

The method of proof is due to Lefmann and Thiele [29]; see also [33]. Denote by $\mathcal{I}(S)$ the number of isosceles triangles spanned by triples of S, where each equilateral triangle is counted three times (this is the same as the number of weighted incidences between perpendicular bisectors determined by S and points of S, where the weight of a bisector is the number of pairs of points for which it is common).

Lemma 3 *(*Lefmann and Thiele [29].) *Let S be a set of n points in the plane, which determine t distinct distances $d_1, d_2, \ldots, d_t$, where d_i occurs with multiplicity m_i for $i = 1, 2, \ldots, t$. Then*

$$\sum_{i=1}^{t} m_i^2 \leq \frac{n}{2} \left(\mathcal{I}(S) + \binom{n}{2} \right).$$

Using the upper bound $\mathcal{I}(S) = O(n^{\alpha+\varepsilon}) = O(n^{2.136})$ from [34], one obtains from Lemma 3:

$$\sum_{i=1}^{t} m_i^2 = O(n^{1+\alpha+\varepsilon}) = O(n^{3.136}). \tag{6}$$

Let $S = \{p_1, p_2, \ldots, p_n\}$ be a set of n points in the plane, which determine t distinct distances $d_1, d_2, \ldots, d_t$, where d_i occurs with multiplicity m_i for $i = 1, 2, \ldots, t$. Now define a hypergraph $\mathcal{H} = (S, \mathcal{E}_3 \cup \mathcal{E}_4)$ as follows:

Let $\{p_i, p_j, p_k\} \in \mathcal{E}_3 \subseteq [S]^3$ if and only if $|p_i - p_j| = |p_i - p_k|$, that is, $\Delta p_i p_j p_k$ is an isosceles triangle. Let $\{p_i, p_j, p_k, p_l\} \in \mathcal{E}_4 \subseteq [S]^4$ if and only if $|p_i - p_j| = |p_k - p_l|$, that is, the two segments have the same length. Let $\varepsilon > 0$ be arbitrary small (but fixed). Clearly, $|\mathcal{E}_3| \leq c_3 \cdot n^{\alpha+\varepsilon}$, and $|\mathcal{E}_4| \leq \sum_{i=1}^{t} \binom{m_i}{2} \leq c_4 \cdot n^{1+\alpha+\varepsilon}$, for some constants $c_3, c_4 > 0$.

To conclude the proof, one makes an experiment consisting of two steps: random sampling followed by the deletion method. In the first step, a random subset $X \subset S$ is chosen by selecting points independently with probability $p = c \cdot n^{-\alpha/3-\varepsilon/3}$, for some constant $c > 0$ to be specified later. In the second step, one point from each "surviving" edge in $\mathcal{E}_3' = [X]^3 \cap \mathcal{E}_3$ and $\mathcal{E}_4' = [X]^4 \cap \mathcal{E}_4$ is deleted. It results in an independent set Y of $\mathcal{H}$ with average size

$$E[|Y|] \geq (c - c^4 c_4) n^{1-\alpha/3-\varepsilon/3} - c^3 c_3,$$

see [29] for details. By choosing c sufficiently small, and after relabeling ε, one gets $E[|Y|] = \Omega(n^{\beta-\varepsilon})$, where $\beta = 1 - \frac{\alpha}{3}$, as claimed.

4 Conclusion

Other variants of the problem of distinct distances—suggested by Erdős—can be obtained by imposing restrictions on the number of distances induced by small point sets [15, 17]. Let $\phi(n, k, l)$ denote the minimum number of distinct distances in a set of n points in the plane with the property that any k points determine at least l distinct distances [6, pp. 203–204]. Erdős conjectured that $\phi(n, 3, 3)$ grows superlinearly, and asked whether $\phi(n, 4, 5) = \Omega(n^2)$ and whether $\phi(n, 5, 9) = \Omega(n^2)$. Not much appears to be known on these variants, and progress has been slow. Some values of ϕ can be deduced from known facts. For instance, the integer lattice determines no equilateral triangle, which implies $\phi(n, 3, 2) = O(n/\sqrt{\log n})$.

As observed by Erdős [15] (and independently in [10]), a classical result of Behrend [5] on the existence of relatively dense sets of integers without any arithmetic progression of length three yields n points on the line with no isosceles triangle, which determine at most $n 2^{c\sqrt{\log n}}$ distinct distances. This implies

$\phi(n, 3, 3) \le n2^{c\sqrt{\log n}}$. A truly "planar" construction with points in general position and no isosceles triangles, that achieves the same bound is given in [18]. For points on the line with no isosceles triangle (i.e., midpoint-free) it is known that the minimum number of distinct distances grows superlinearly [10], however this seems to be of little relevance for the general planar case.

Erdős also noted [15] that $\phi(n, 6, 14) \ge \frac{1}{2}\binom{n}{2}$, since the same distance can occur at most twice. To put these variants in a better context, we further observe that $\phi(n, 4, 4) \le n2^{c\sqrt{\log n}}$. To see this, recall Behrend's construction mentioned earlier, i.e., a midpoint-free n-element point set on the line. Observe that any four points determine at least four distinct distances, since any of the six distances can occur at most twice, except the maximum distance which is unique. Of course the construction can be also made "planar" (with no three collinear points) if desired, by placing the points on a small circular arc, so that the distance structure is preserved. Finally notice that neither of the above two estimates ($\phi(n, 3, 3) \le n2^{c\sqrt{\log n}}$ and $\phi(n, 4, 4) \le n2^{c\sqrt{\log n}}$) implies the other. (i) Four points that determine no isosceles triangle, do not necessarily determine four distinct distances; take for example the four vertices of a generic rectangle, which determine only three distinct distances. (ii) In the other direction, it is easy to exhibit point sets containing isosceles triangles in which every 4-tuple determines at least four distinct distances.

We conclude with the following (hopefully easier) question: Does there exist a set S of n points in the plane in general position, that are also in convex position, so that $g(S)$, the number of distinct distances determined by S, is $o(n^2)$? If the answer is positive, estimate the minimum number of distances as best as possible.

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