Covering a disk by disks

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Abstract

For a convex body C in $\mathbb{R}^d$, what is the smallest number $f = f_d(C)$ such that any sequence of smaller homothetic copies of C whose total volume is at least f times the volume of C permits a translative covering of C? László Fejes Tóth conjectured in 1984 that $f_2(C) \leq 3$ for any convex body C in the plane. This conjecture has been only confirmed for parallelograms and triangles: Moon and Moser had shown in 1967 that $f_2(C) = 3$ for a square C. Since $f_d(C)$ is invariant under affine transformation of C, it follows from Moon and Moser's result that $f_2(C) = 3$ for any parallelogram C. In 2003, Füredi settled the case of triangles with a sharper bound, by showing that $f_2(C) = 2$ for a triangle C, and thus confirming a stronger conjecture of A. Bezdek and K. Bezdek for this case. For an arbitrary planar convex body C, the current best bound is $f_2(C) \leq 6.5$, due to Januszewski. In this paper, we prove that $f_2(D) < 3$ for a disk D, and thereby confirm the conjecture of László Fejes Tóth for disks. We also present the first non-trivial bound for covering a disk by disks in the online model. Our methods lead to very efficient algorithms for both offline and online disk covering.

1 Introduction

Covering a convex body by its smaller homothetic copies is a classic problem in geometry, that has generated a lot of interest over the years, initially in finding optimal structural patterns for packing and covering, and more recently in designing efficient online algorithms.

Let C be a convex body in $\mathbb{R}^d$, that is, a compact convex set with nonempty interior in the d-dimensional Euclidean space. Let $\mathcal{C} = \{C_1, C_2, \ldots\}$ be a (possibly infinite) sequence of convex bodies in $\mathbb{R}^d$. We say that $\mathcal{C}$ permits a *covering* of C if there exist rigid motions σ_i such that $C \subseteq \bigcup_i \sigma_i(C_i)$. We say that $\mathcal{C}$ permits a *translative covering* of C if there exist translations τ_i such that $C \subseteq \bigcup_i \tau_i(C_i)$. Define $f_d(C)$ as the smallest number f with the following property:

> Any sequence of smaller homothetic copies of C whose total volume is at least f times the volume of C permits a translative covering of C.

Very recently, Naszódi [14] showed that $f_d(C) \leq 6^d$ for any convex body C in $\mathbb{R}^d$; for the planar case, the current best bound, $f_2(C) \leq 6.5$, is due to Januszewski [7]. It is also known that $f_2(C) = 3$ for a square C (indeed for any parallelogram C too since $f_d(C)$ is invariant under affine transformation of C) [13] and that $f_2(C) = 2$ for a triangle C [3].

László Fejes Tóth conjectured [1, Page 131, Conjecture 1] in 1984 that

> *For any planar convex body C, $f_2(C) \leq 3$.*

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Besides this offline setting, the above problem has also been studied in the online model [11, 8, 9, 10, 12], where each homothetic copy in the sequence is revealed only after the placement of the preceding copy in the sequence. Define $g_d(C)$ for the online model, analogous to $f_d(C)$ for the offline model. Obviously $f_d(C) \leq g_d(C)$ holds for any C and d. It is known that $g_2(C) \leq 28$ for any convex body C [8][1]. It is also known that $g_2(C) \leq \frac{7}{4}\sqrt[3]{9} + \frac{13}{8} = 5.265\ldots$ for a square C [9], $g_3(C) \leq 9.843\ldots$ for a cube C [12], and $g_d(C) \leq 2^d + (5/3)(1 + 2^{-d})$ for a hypercube C in $\mathbb{R}^d$ [12]. For a survey of this and many other related problems, we refer the reader to the book by Braß et al. [1, Chapter 3].

All previous results on $f_2(C)$ and $g_2(C)$, except the two general results $f_2(C) \leq 6.5$ [7] and $g_2(C) \leq 28$ [8] for any convex body in the plane, are for simple convex bodies with straight boundaries, such as squares and triangles. While the conjecture of László Fejes Tóth that $f_2(C) \leq 3$ for any planar convex body has been confirmed for squares [13] and triangles [3], it was not confirmed until now for any natural convex body with a curved boundary, where the analysis is more difficult. In this paper, we prove the conjecture of László Fejes Tóth for disks, the most natural convex bodies with curved boundaries.

Let D be a unit disk[2]. Write $\rho = f_2(D)$. That is, ρ is the smallest number such that any sequence of disks with total area at least ρ times the area of a unit disk permits an offline covering of the unit disk (note that any disk covering is automatically translative). For $k \geq 1$, also define ρ_k, similar to ρ, but with the additional constraint that the sequence contains *at most* k disks. It is clear that $\rho \geq \rho_{k+1} \geq \rho_k$ for any $k \geq 1$, and that $\rho = \lim_{k\to\infty} \rho_k$. Write $\eta = g_2(D)$ for the online model.

Our main results are summarized in the following three theorems. Theorem 1 and Theorem 3 are both obtained by analytical proofs, while Theorem 2 is obtained by a computer-assisted proof.

Theorem 1. *Any sequence of disks with total area at least* 3.25 *times the area of a unit disk permits an offline covering of the unit disk. That is,* $\rho \leq 3.25$*. Moreover,* $\rho_1 = 1$*,* $\rho_2 = 2$*, and* $\rho_3 = \rho_4 = 2.25$*.*

Theorem 2. *Any sequence of disks with total area at least* 2.97 *times the area of a unit disk permits an offline covering of the unit disk. That is,* $\rho \leq 2.97$*.*

Theorem 3. *Any sequence of disks with total area at least* 9.7633 *times the area of a unit disk permits an online covering of the unit disk. That is,* $\eta \leq 9.7633$*.*

Note that Theorem 2 confirms the conjecture of László Fejes Tóth for the disk case. Our methods for obtaining these bounds are constructive and lead to very efficient algorithms for disk covering. In particular, Theorems 1 and 2 lead to $O(n)$ time algorithms for offline covering a unit disk by a sequence of n disks ordered by non-increasing radius, and Theorem 3 leads to an $O(n \log n)$ time algorithm for online covering a unit disk by a sequence of n disks.

In this paper, we cover the unit disk D by a sequence $\mathcal{D} = \langle D_1, D_2, \ldots \rangle$ of disks. The sequence $\mathcal{D}$ can be finite or infinite. If the sequence is finite and a reference is made to a disk D_i whose index i is larger than the total number of disks, we assume for convenience that D_i exists but has zero radius. While for offline covering we will assume that the disks in the input sequence are sorted by radius (largest disk first), no such assumption is made for online covering.

2 Offline covering

In this section we prove Theorems 1 and 2. Let the *unit disk* D be a disk of unit *radius*. Denote by x_i the radius of the ith disk D_i in the sequence $\mathcal{D}$. Assume that $1 > x_1 \geq x_2 \geq \cdots$. The largest four or five disks in

[1] A method by Januszewski [5] achieves a bound of 15, but the covering is not translative because it uses 90° rotations. Braß et al. [1, Page 126, Problem 1] incorrectly state that 15 is the current best bound for translative covering.

[2] Whether D has unit radius or unit diameter will be made clear in the respective sections. For convenience, we will use different definitions of unit disk in our analysis for $f_2(D)$ and $g_2(D)$; the bounds are not affected by this difference because they are ratios.

the sequence play especially important roles in our proofs because they will be used first, in a greedy manner, to cover either the whole unit disk or two large cap regions of it. To simplify the case analysis for the relative disk sizes in our proofs, we will transform the largest four disks while maintaining the non-increasing order of the disk radii in the sequence.

2.1 Proof of Theorem 1

We first introduce two covering tools, next give bounds for the special case of covering the unit disk by at most four disks, and then prove a general bound of $\rho \leq 3.25$.

2.1.1 Covering a rectangle by disks

Define the following function A of three variables $w, h, x \in \mathbb{R}$:

$$A(w,h,x) = \min\left\{w(h+x) + hx, w(h+x) + h^2\right\}.$$

For $w \geq h > 0$ and $x > 0$, $A(w,h,x)$ is an area measure used in the following lemma by Januszewski [6], which is an extension of the classical result by Moon and Moser [13] on translative coverings of the unit square by smaller homothetic squares.

Lemma 1. (Januszewski [6]). *Given an axis-parallel rectangle with width w and height h ($h \leq w$) and a sequence of axis-parallel squares with side length at most x, if the total area of the squares is at least $A(w,h,x)$, then the rectangle permits a translative covering by the squares.*

Observe that a disk D_k of radius x_k contains a square of side length $\sqrt{2}\,x_k$ in any orientation. This leads to the following corollary:

Corollary 1. *Given a rectangle with width w and height h ($h \leq w$) and a sequence of disks with radius at most x, if the total area of the disks is at least $\frac{\pi}{2}A(w,h,\sqrt{2}\,x)$, then the rectangle can be covered by the disks.*

2.1.2 Covering a cap by two disks

For any $i \geq 1$, put $a_i = \arcsin x_i$. For any $i \neq j$, put $h_{ij} = \cos(a_i + a_j)$. We first prove the following lemma:

Lemma 2. *For any $i \neq j$, D_i and D_j can be placed to cover a cap of angle $2a_i + 2a_j$ and height $1 - h_{ij}$ of the unit disk D.*

Proof. Refer to Figure 1, where the shaded triangle $\triangle spq$ is inscribed in the unit disk, and has two sides sp and sq of lengths $2x_i$ and $2x_j$, respectively. Place D_i and D_j such that the two sides sp and sq are their diameters. Then the third side pq of the triangle intersects the boundaries of both D_i and D_j at exactly the same point t, where $st \perp pq$. D_i and D_j together cover a cap of the unit disk bounded by pq, which subtends an angle of $2a_i + 2a_j$ from the unit disk center. The height of the cap is $1 - h_{ij}$, where $h_{ij} = \cos(a_i + a_j)$ is the signed distance from pq to the unit disk center (h_{ij} is negative if $2a_i + 2a_j > \pi$; in this case the unit disk center lies inside the triangle $\triangle spq$). $\square$

We now prove some useful properties for covering a cap by two disks:

Lemma 3. *Let $i < j$ (thus $x_i \geq x_j$ and $a_i \geq a_j$). Suppose that $0 \leq 2a_j \leq 2a_i \leq \pi$ and that $2a_i + 2a_j$ is fixed (thus h_{ij} is also fixed). Then we have:*

(i) *If $2a_i + 2a_j \leq \pi$ ($h_{ij} \geq 0$), then $x_i^2 + x_j^2$ is non-decreasing when x_i increases and correspondingly x_j decreases.*

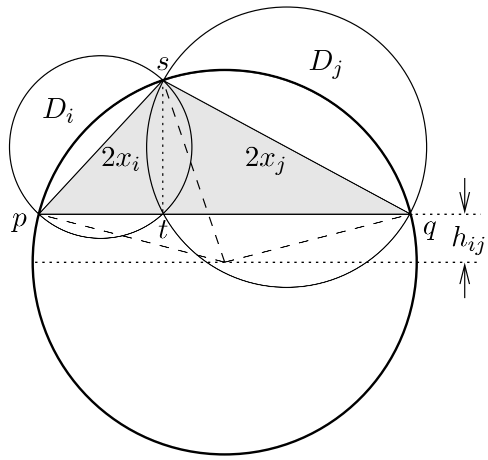

Figure 1: Covering a cap by two disks.

(ii) *If* $2a_i + 2a_j \geq \pi$ *(*$h_{ij} \leq 0$*), then* $x_i^2 + x_j^2$ *is non-decreasing when* x_i *decreases and correspondingly* x_j *increases until* $x_i = x_j$*.*

Proof. Since $2a_i + 2a_j$ is fixed, we have $\frac{\mathrm{d}a_j}{\mathrm{d}a_i} = -1$. Then,

$$\frac{\mathrm{d}(x_i^2 + x_j^2)}{\mathrm{d}a_i} = \frac{\mathrm{d}(\sin^2 a_i + \sin^2 a_j)}{\mathrm{d}a_i} = \sin 2a_i - \sin 2a_j.$$

Consider two cases:

(i) $2a_i + 2a_j \leq \pi$ ($h_{ij} \geq 0$).
If $0 \leq 2a_i \leq \pi/2$, then $0 \leq 2a_j \leq 2a_i \leq \pi/2$. If $\pi/2 \leq 2a_i \leq \pi$, then $0 \leq 2a_j \leq \pi - 2a_i \leq \pi/2$. We always have $\sin 2a_i - \sin 2a_j \geq 0$. Therefore $x_i^2 + x_j^2$ is non-decreasing when a_i (hence x_i) increases and correspondingly a_j (hence x_j) decreases.

(ii) $2a_i + 2a_j \geq \pi$ ($h_{ij} \leq 0$).
If $0 \leq 2a_j \leq \pi/2$, then $0 \leq \pi - 2a_i \leq 2a_j \leq \pi/2$. If $\pi/2 \leq 2a_j \leq \pi$, then $\pi/2 \leq 2a_j \leq 2a_i \leq \pi$. We always have $\sin 2a_i - \sin 2a_j \leq 0$. Therefore $x_i^2 + x_j^2$ is non-decreasing when a_i (hence x_i) decreases and correspondingly a_j (hence x_j) increases.

The intersection of the two cases is: $2a_i + 2a_j = \pi$ ($h_{ij} = 0$). In this case, $2a_i = \pi - 2a_j$, $\sin 2a_i - \sin 2a_j = 0$, and $x_i^2 + x_j^2$ is constant. □

2.1.3 The special case of covering the unit disk by at most four disks

It is a simple exercise to prove that $\rho_1 = 1$ and $\rho_2 = 2$. To show that $\rho_3 = \rho_4 = 2.25$, we first prove the following lemma:

Lemma 4. *If* $2a_1 + 2a_2 + 2a_3 + 2a_4 < 2\pi$*, then* $x_1^2 + x_2^2 + x_3^2 + x_4^2 < 2.25$*.*

Proof. Consider four disks D_1, D_2, D_3, and D_4 with a fixed value of $2a_1 + 2a_2 + 2a_3 + 2a_4$ such that $x_1^2 + x_2^2 + x_3^2 + x_4^2$ is maximized. Recall that $1 > x_1 \geq x_2 \geq x_3 \geq x_4$. Since $2a_1 + 2a_2 + 2a_3 + 2a_4 < 2\pi$, we must have $2a_3 + 2a_4 < \pi$. By Lemma 3(i), we can increase x_3 and decrease x_4 with $2a_3 + 2a_4$ fixed until either $x_2 = x_3$ or $x_4 = 0$. If $x_4 > 0$ after this transformation, then $2a_2 + 2a_4 = 2a_3 + 2a_4 < \pi$, and again by Lemma 3(i) we can increase x_2 and decrease x_4 until either $x_1 = x_2$ or $x_4 = 0$. If we still have

$x_4 > 0$, then perform one more such transformation to increase x_1 and decrease x_4 until $x_4 = 0$. Finally, we have $a_1 \geq a_2 \geq a_3 \geq a_4 = 0$ and $a_1 + a_2 + a_3 < \pi$. Since $x_i = \sin^2 a_i$ is an increasing function of a_i for $0 \leq a_i \leq \pi/2$, we can find three angles α, β, and γ of a triangle such that $a_1 \leq \alpha$, $a_2 \leq \beta$, $a_3 \leq \gamma$, and $\sin^2 a_1 + \sin^2 a_2 + \sin^2 a_3 < \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma$. Then $x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_1^2 + x_2^2 + x_3^2 = \sin^2 a_1 + \sin^2 a_2 + \sin^2 a_3 < \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma \leq 2.25$, where the last step is a well-known inequality in the geometry of triangles [16]. □

If $x_1^2 + x_2^2 + x_3^2 + x_4^2 \geq 2.25$, then it follows by Lemma 4 that $2a_1 + 2a_2 + 2a_3 + 2a_4 \geq 2\pi$. Then, by Lemma 2, the two pairs of disks (D_1, D_2) and (D_3, D_4) can be placed to cover two caps whose union is the unit disk. Therefore $\rho_4 \leq 2.25$. On the other hand, it is easy to see that $\rho_3 \geq 2.25$, as given by the configuration of three equal disks whose diameters form an equilateral triangle inscribed in the unit disk. In fact, Füredi [4] conjectured that this configuration is the overall worst case and that $\rho = 2.25$. Since $\rho_4 \geq \rho_3$, we obtain the tight bounds $\rho_3 = \rho_4 = 2.25$. This proves the second part of Theorem 1.

2.1.4 A general bound of $\rho \leq 3.25$

We now prove the first part of Theorem 1 by considering four cases (it is easy to check that they cover all possibilities):

Case 0 that $2a_1 + 2a_2 + 2a_3 + 2a_4 \geq 2\pi$:
D_1, D_2, D_3, and D_4 together cover the whole unit disk D.

Case 1 that $2a_1 + 2a_2 \geq \pi$ and $2a_3 < \pi/2$:
D_1 and D_2 together cover a half of D; D_3 covers less than a quarter of D.

Case 2 that $2a_1 \geq 2a_2 \geq 2a_3 \geq \pi/2$:
D_1, D_2, and D_3 together cover three quarters of D.

Case 3 that $2a_1 + 2a_2 < \pi$:
D_1 and D_2 together cover less than a half of D.

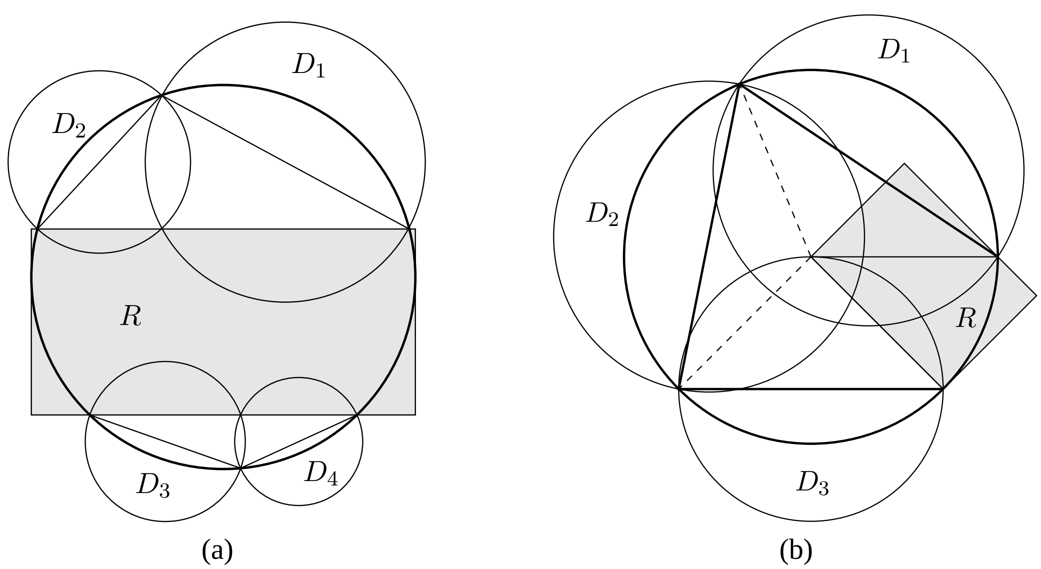

Figure 2: (a) Two caps and a rectangle. (b) Three sectors and a rectangle.

For Cases 0, 1, and 3, we use our main method illustrated in Figure 2(a): D_1 and D_2 cover a cap of height $1-h_{12}$; D_3 and D_4 cover a cap of height $1-h_{34}$, so that the chords of the two caps are parallel. The other disks cover a rectangle R of width 2 and height $h_{12}+h_{34}$ (if $h_{12}+h_{34}>0$).

For Case 2, we use an alternative method illustrated in Figure 2(b): D_1, D_2, and D_3 cover three sectors of angles $2a_1$, $2a_2$, and $2a_3$, so that the diameters of the three disks are consecutive chords of the unit disk D. The other disks cover a rectangle R of width 1 and height $\sin(2\pi-2a_1-2a_2-2a_3)$.

In all four cases, we use a few large disks $D_1,\ldots,D_k$, $k=3$ or 4, to cover part of the unit disk, then use the remaining small disks D_i, $i\geq k+1$, to cover a rectangle R of width w and height h. By Corollary 1, the rectangle R can be covered by the remaining disks if their total area is at least $\frac{\pi}{2}A(w,h,\sqrt{2}\,x_{k+1})$. Define

$$r=\frac{\pi x_1^2+\ldots+\pi x_k^2+\frac{\pi}{2}A(w,h,\sqrt{2}\,x_{k+1})}{\pi}=x_1^2+\ldots+x_k^2+A(w,h,\sqrt{2}\,x_{k+1})/2. \tag{1}$$

Then $\rho\leq r$. It remains to bound the value of r in each of the four cases.

Case 0 that $2a_1+2a_2+2a_3+2a_4\geq 2\pi$

Refer to Figure 1. D_1 and D_2 cover a cap of angle $2a_1+2a_2$, while D_3 and D_4 cover a (parallel) cap of angle $2a_3+2a_4$. Since $2a_1+2a_2+2a_3+2a_4\geq 2\pi$, the two caps overlap hence together they cover the unit disk D. We can assume without loss of generality that $x_5=0$, and suppose that $\sum_{i=1}^{\infty}x_i^2=\sum_{i=1}^{4}x_i^2\geq 2.25$. Then Lemma 4 implies a bound of $\rho\leq 2.25<3.25$ for Case 0.

Case 1 that $2a_1+2a_2\geq\pi$ and $2a_3<\pi/2$

We will reduce Case 1 to either Case 2 or Case 3. By Lemma 3(ii), we can assume that $a_1=a_2$. Then,

$$x_1^2+x_2^2=\sin^2 a_1+\sin^2 a_2=2\sin^2 a_1=1-\cos 2a_1=1-\cos(a_1+a_2)=1-h_{12},$$

$$\frac{\mathrm{d}(x_1^2+x_2^2)}{\mathrm{d}h_{12}}=-1.$$

Fix x_4. Then,

$$\begin{aligned}
x_3^2+x_4^2=\sin^2 a_3+\sin^2 a_4 \quad&\Longrightarrow\quad \frac{\mathrm{d}(x_3^2+x_4^2)}{\mathrm{d}a_3}=\sin 2a_3,\\
h_{34}=\cos(a_3+a_4) \quad&\Longrightarrow\quad \frac{\mathrm{d}a_3}{\mathrm{d}h_{34}}=-\frac{1}{\sin(a_3+a_4)},\\
0\leq a_3+a_4\leq 2a_3<\pi/2 \quad&\Longrightarrow\quad 0\leq\sin(a_3+a_4)\leq\sin(2a_3)\leq 1,
\end{aligned}$$

$$\frac{\mathrm{d}(x_3^2+x_4^2)}{\mathrm{d}h_{34}}=\frac{\mathrm{d}(x_3^2+x_4^2)}{\mathrm{d}a_3}\cdot\frac{\mathrm{d}a_3}{\mathrm{d}h_{34}}=-\frac{\sin 2a_3}{\sin(a_3+a_4)}\leq -1.$$

Therefore,

$$\frac{\mathrm{d}(x_3^2+x_4^2)}{\mathrm{d}h_{34}}\leq\frac{\mathrm{d}(x_1^2+x_2^2)}{\mathrm{d}h_{12}},$$

which implies that, when keeping $h_{12}+h_{34}$ fixed, the sum $x_1^2+x_2^2+x_3^2+x_4^2$ does not decrease if we decrease h_{34} (increase x_3 and fix x_4) and correspondingly increase h_{12} (decrease x_1 and x_2 together). Since

$$2a_1=2a_2\geq\pi/2>2a_3,$$

as we decrease $2a_1 = 2a_2$, and correspondingly increase $2a_3$, either $2a_3$ will become larger than $\pi/2$, or $2a_1 = 2a_2$ will become smaller than $\pi/2$. Case 1 is therefore reduced to either Case 2 that $2a_1 \geq 2a_2 \geq 2a_3 \geq \pi/2$ or Case 3 that $2a_1 + 2a_2 < \pi$.

Case 2 that $2a_1 \geq 2a_2 \geq 2a_3 \geq \pi/2$

D_1, D_2, and D_3 cover three sectors of angles $2a_1$, $2a_2$, and $2a_3$ of the unit disk. Put $\theta = 2\pi - (2a_1 + 2a_2 + 2a_3)$. If $\theta \leq 0$, then we would have Case 0 that $2a_1 + 2a_2 + 2a_3 + 2a_4 \geq 2\pi$. So assume that $0 < \theta \leq \pi/2$. The sector of angle θ is contained in a rectangle R of width $w = 1$ and height $h = \sin\theta$; see Figure 2(b).

By Lemma 3(ii), we can assume that $a_1 = a_2 = a_3 = (2\pi - \theta)/6$. Therefore,

$$x_1^2 + x_2^2 + x_3^2 = 3\sin^2\frac{2\pi - \theta}{6} = \frac{3}{2}\left(1 - \cos\frac{2\pi - \theta}{3}\right) = \frac{3}{2}\left(1 + \cos\frac{\theta + \pi}{3}\right).$$

If $2a_4 \geq \pi/2$, then we would again have Case 0 that $2a_1 + 2a_2 + 2a_3 + 2a_4 \geq 2\pi$. So assume otherwise. Then $x_4 < \sqrt{2}/2$, and we have

$$A(w, h, \sqrt{2}\,x_4)/2 \leq A(1, \sin\theta, 1)/2 \leq \big(1 \cdot (\sin\theta + 1) + \sin\theta \cdot 1\big)/2 = 1/2 + \sin\theta = 1/2 - \sin(\theta + \pi).$$

Therefore,

$$r = x_1^2 + x_2^2 + x_3^2 + A(w, h, \sqrt{2}\,x_4)/2 \leq 2 + \frac{3}{2}\cos\frac{\theta + \pi}{3} - \sin(\theta + \pi).$$

Put $\gamma = (\theta + \pi)/3$. Then

$$r = r(\gamma) = 2 + (3/2)\cos\gamma - \sin 3\gamma, \qquad \pi/3 < \gamma \leq \pi/2. \tag{2}$$

Setting $\frac{\mathrm{d}r(\gamma)}{\mathrm{d}\gamma}$ to zero to maximize $r(\gamma)$, we have

$$\begin{aligned} -(3/2)\sin\gamma - 3\cos 3\gamma &= 0 \\ \sin^2\gamma &= 4\cos^2 3\gamma \\ 1 - \cos^2\gamma &= 4(4\cos^2\gamma - 3)^2\cos^2\gamma. \end{aligned}$$

Put $x = \cos^2\gamma$, and get the following cubic equation:

$$64x^3 - 96x^2 + 37x - 1 = 0, \qquad 0 < x \leq 1/4. \tag{3}$$

Equation (3) has only one real root $0.02919\ldots$ between 0 and $1/4$. Correspondingly, $r(\gamma)$ attains the maximum value $3.126\ldots$ at $\gamma = 80.16\ldots^\circ$. We have obtained a bound of $\rho \leq 3.126\ldots < 3.25$ for Case 2.

Case 3 that $2a_1 + 2a_2 < \pi$

The condition implies $h_{12} > 0$. Likewise we also have $2a_3 + 2a_4 < \pi$, and correspondingly $h_{34} > 0$. Since $2a_3 + 2a_4 < \pi$, we have $a_4 < \pi/4$. D_1, D_2, D_3, and D_4 cover two caps of total height $2 - h_{12} - h_{34}$. The remaining uncovered area of the unit disk is contained in a rectangle of width $w = 2$ and height $h = h_{12} + h_{34}$.

For two variables x and y, $0 \leq x \leq y \leq \pi/2$, such that $x + y$ is fixed, we have

$$\frac{\mathrm{d}(\cos x + \cos y)}{\mathrm{d}x} = -\sin x + \sin y \geq 0.$$

Therefore,

$$h = h_{12} + h_{34} = \cos(a_1 + a_2) + \cos(a_3 + a_4) \leq \cos(a_1 + a_4) + \cos(a_2 + a_3). \quad (4)$$

Note that $a_5 \leq a_4 < \pi/4$ hence $x_5 < \sqrt{2}/2$. We have

$$A(w, h, \sqrt{2}\, x_5)/2 \leq h + \sqrt{2}\, x_5 + (h\sqrt{2}\, x_5)/2 < h + \sqrt{2}\, x_5 + h/2 \leq 1.5h + \sqrt{2}\, x_4. \quad (5)$$

By Lemma 3(i), we can enlarge D_1 and correspondingly shrink D_2 until $x_2 = x_3$. So assume that $a_2 = a_3$. Now it follows from (5) and (4) that

$$\begin{aligned} r &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + A(2, h, \sqrt{2}\, x_5)/2 \\ &< \sin^2 a_1 + 2\sin^2 a_2 + \sin^2 a_4 + 1.5(\cos(a_1 + a_4) + \cos 2a_2) + \sqrt{2}\sin a_4 \\ &= \sin^2 a_1 + 1.5\cos(a_1 + a_4) + \sqrt{2}\sin a_4 + \sin^2 a_4 + 2\sin^2 a_2 + 1.5\cos 2a_2 \\ &= \sin^2 a_1 + 1.5\cos(a_1 + a_4) + \sqrt{2}\sin a_4 + \sin^2 a_4 - \sin^2 a_2 + 1.5 \\ &\leq \sin^2 a_1 + 1.5\cos(a_1 + a_4) + \sqrt{2}\sin a_4 + 1.5. \end{aligned}$$

Fix $a_1 + a_4$. Then,

$$\frac{\mathrm{d}r}{\mathrm{d}a_1} = \sin 2a_1 - \sqrt{2}\cos a_4 < \sin 2a_1 - 1 \leq 0,$$

where the first inequality follows from $a_4 < \pi/4$. Therefore we can assume that $a_1 = a_4$.

Put $\theta = a_1 = a_4$. Then,

$$\begin{aligned} r &< \sin^2\theta + 1.5\cos 2\theta + \sqrt{2}\sin\theta + 1.5 \\ &= -2\sin^2\theta + \sqrt{2}\sin\theta + 3 \\ &= -2(\sin\theta - \sqrt{2}/4)^2 + 3.25 \\ &\leq 3.25. \end{aligned}$$

We have obtained a bound of $\rho \leq 3.25$ for Case 3, and the proof of Theorem 1 is now complete.

Extremal configuration It is illuminating to take a closer look at the extremal configuration in Case 3 for the 3.25 bound, where $x_1 = x_2 = x_3 = x_4 = x_5 = \sqrt{2}/4 = 0.3535\ldots$. A calculation shows that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.5$, $h = 2\cos 2a_2 = 2(1 - 2\sin^2 a_2) = 2(1 - 1/4) = 1.5$, and

$$A(w, h, \sqrt{2}\, x_5)/2 = A(2, 1.5, 0.5)/2 = 1.5 + 0.5 + 0.375 = 2.375.$$

So the real bound for the extremal configuration should be

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + A(w, h, \sqrt{2}\, x_5)/2 = 0.5 + 2.375 = 2.875.$$

However, to simplify the analysis, we have used a rather conservative estimate $x_5 < \sqrt{2}/2 = 0.7071\ldots$ in the second inequality in (5), which led to a looser bound:

$$A(w, h, \sqrt{2}\, x_5)/2 < 1.5 + 0.5 + 0.75 = 2.75,$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + A(w, h, \sqrt{2}\, x_5)/2 < 0.5 + 2.75 = 3.25.$$

We found it difficult to obtain a better bound than 3.25 with an analytical proof (note that it is not trivial even to determine the minimum radius of five equal disks that cover a unit disk [15]). However with the help of a computer program we have obtained a bound less than 3 (in the next subsection).

2.2 Proof of Theorem 2

We will use two more covering tools to obtain a bound of $\rho \leq 2.97$ with a computer-assisted proof.

2.2.1 Two more covering tools

Let D_i, D_j, D_k be three disks such that $i < j < k$ (thus $x_i \geq x_j \geq x_k$). Note that D_j contains a copy of D_k. Refer to Figure 3, where the shaded trapezoid is inscribed in the unit disk. Place the large disk D_i and two copies of the small disk D_k such that (i) the centers of the three disks are collinear, (ii) the diameters of the two copies of D_k are the left and right sides of the trapezoid, and (iii) the boundary of D_i passes through the two vertices of the upper side of the trapezoid. Then the lower side of the trapezoid intersects the boundaries of D_i and each copy of D_k at exactly the same point. The three disks together cover a cap of the unit disk.

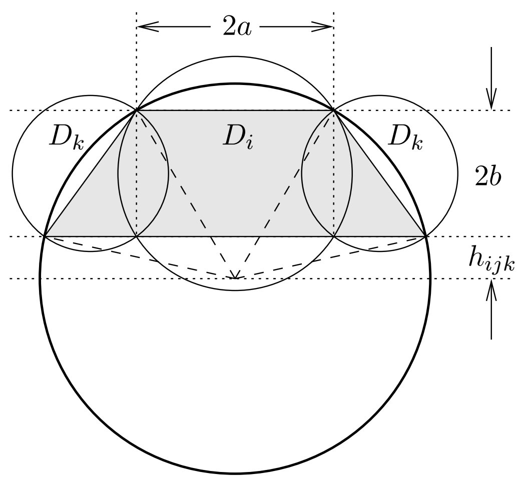

Figure 3: Covering a cap by one large disk and two equal small disks.

Let h_{ijk} be the signed distance from the lower side of the trapezoid to the unit disk center. Let $2a$ and $2b$, respectively, be the upper side length and the height of the trapezoid. Let 2α and 2β, respectively, be the two angles subtended by the lower and upper sides of the trapezoid from the unit disk center. The five parameters h_{ijk}, a, b, α, and β are determined by x_i and x_k according to the following five equations:

$$\cos\alpha = h_{ijk}, \quad \cos\beta = h_{ijk} + 2b, \quad \alpha - \beta = 2\arcsin x_k, \quad a = \sin\beta, \quad a^2 + b^2 = x_i^2.$$

We have the following lemma by construction:

Lemma 5. *D_i, D_j, and D_k can be placed to cover a cap of height $1 - h_{ijk}$ of the unit disk.*

We will also use the following lemma by Neville [15] which provides the solution to a popular problem from the 19th century[3]:

Lemma 6. (Neville [15]). *A unit disk can be covered by five equal disks of radius $0.609383\ldots$.*

[3][2, Problem D3]: "The problem of completely covering a circular region by placing over it, one at a time, five smaller equal circular disks was familiar to frequenters of English fairs a century ago."

2.2.2 A bound of $\rho \leq 2.97$ with a computer-assisted proof

Put $\hat{x}_5 = 0.6094$ and $\hat{r} = 2.97$. Recall our definition of r in (1). Now define r_4 and r_5 as follows:

$$r_4 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + A(2, h_{12} + h_{34}, \sqrt{2}\, x_5)/2, \tag{6}$$
$$r_5 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + A(2, h_{12} + h_{345}, \sqrt{2}\, x_5)/2. \tag{7}$$

Our next two lemmas are about the four conditions $h_{12} + h_{34} \leq 0$, $x_5 \geq \hat{x}_5$, $\hat{r} \geq r_4$, and $\hat{r} \geq r_5$.

Lemma 7. *If the total area of the disks in $\mathcal{D}$ is at least $\hat{r}$ times the area of the unit disk D, and if any one of the four conditions is satisfied, then the unit disk D can be covered by the disks in $\mathcal{D}$.*

Proof. We give a covering method for each condition:

1. $h_{12} + h_{34} \leq 0$: By Lemma 2, the unit disk can be covered as follows:
   (a) D_1 and D_2 cover a cap of height $1 - h_{12}$;
   (b) D_3 and D_4 cover a cap of height $1 - h_{34}$.
2. $x_5 \geq \hat{x}_5$: By Lemma 6, the unit disk can be covered by the five disks D_1, D_2, D_3, D_4, and D_5.
3. $\hat{r} \geq r_4$: By Lemma 2 and Corollary 1, the unit disk can be covered as follows:
   (a) D_1 and D_2 cover a cap of height $1 - h_{12}$;
   (b) D_3 and D_4 cover a cap of height $1 - h_{34}$;
   (c) If $h_{12} + h_{34} > 0$, the other disks cover a rectangle of width 2 and height $h_{12} + h_{34}$.
4. $\hat{r} \geq r_5$: By Lemma 2, Lemma 5, and Corollary 1, the unit disk can be covered as follows:
   (a) D_1 and D_2 cover a cap of height $1 - h_{12}$;
   (b) D_3, D_4, and D_5 cover a cap of height $1 - h_{345}$;
   (c) If $h_{12} + h_{345} > 0$, the other disks cover a rectangle of width 2 and height $h_{12} + h_{345}$. □

Lemma 8. *If the total area of the disks in $\mathcal{D}$ is at least $\hat{r}$ times the area of the unit disk D, then at least one of the four conditions is satisfied.*

Proof. We were unable to find a simple analytical proof of Lemma 8, but have verified it by a computer program (Appendix A). The program enumerates all discrete combinations of $(x_1, x_2, x_3, x_4, x_5)$ where $1 > x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5 \geq 0$ with the step size[4] $\delta = 0.005$. For each discrete combination $(x_1, x_2, x_3, x_4, x_5)$, the program uses closed-form formulas to calculate

$$h_{12} = \cos(\arcsin(x_1) + \arcsin(x_2))$$
$$h_{34} = \cos(\arcsin(x_3) + \arcsin(x_4)),$$

and uses a binary search to find a value $\hat{h}_{345}$ such that $h_{345}(x_3, x_4, x_5) \leq \hat{h}_{345}$. To account for the sampling error, the program uses the enlarged values $x_i + \delta$ instead of x_i in (6) and (7) to calculate

$$\hat{r}_4 = (x_1 + \delta)^2 + (x_2 + \delta)^2 + (x_3 + \delta)^2 + (x_4 + \delta)^2 + A(2, h_{12} + h_{34}, \sqrt{2}\,(x_5 + \delta))/2$$
$$\hat{r}_5 = (x_1 + \delta)^2 + (x_2 + \delta)^2 + (x_3 + \delta)^2 + (x_4 + \delta)^2 + (x_5 + \delta)^2 + A(2, h_{12} + \hat{h}_{345}, \sqrt{2}\,(x_5 + \delta))/2.$$

[4] With the step size $\delta = 0.005$, the program takes less than one minute on a low-end desktop computer (tested on an Apple iMac computer with a 2GHz PowerPC G5 processor running Mac OS X 10.4.11). A smaller step size (with a longer running time) gives a bound better than 2.97, but not below 2.9.

The program then verifies that at least one of the four conditions is satisfied.

Note that h_{12} is a decreasing function of x_1 and x_2, and h_{34} is a decreasing function of x_3 and x_4. Although we don't have a closed-form formula for h_{345}, it is clear from our construction in Figure 3 that h_{345} is a non-increasing function of x_3, x_4, and x_5. Also note that $A(w,h,x)$ is a non-decreasing function of w, h, and x. Therefore, for any (not necessarily discrete) combination $(x'_1, x'_2, x'_3, x'_4, x'_5)$ such that $x_i \le x'_i \le x_i + \delta$, $1 \le i \le 5$, we have

$$\begin{aligned} h_{12}(x_1,x_2) + h_{34}(x_3,x_4) \le 0 &\quad\Longrightarrow\quad h_{12}(x'_1,x'_2) + h_{34}(x'_3,x'_4) \le 0 \\ x_5 \ge \hat{x}_5 &\quad\Longrightarrow\quad x'_5 \ge \hat{x}_5 \\ \hat{r} \ge \hat{r}_4(x_1,x_2,x_3,x_4,x_5) &\quad\Longrightarrow\quad \hat{r} \ge r_4(x'_1,x'_2,x'_3,x'_4,x'_5) \\ \hat{r} \ge \hat{r}_5(x_1,x_2,x_3,x_4,x_5) &\quad\Longrightarrow\quad \hat{r} \ge r_5(x'_1,x'_2,x'_3,x'_4,x'_5). \end{aligned}$$

Since all discrete combinations are checked by the program, it follows that all possible combinations are also verified. □

By Lemma 7 and Lemma 8, it follows that $\rho \le \hat{r} = 2.97$. The proof of Theorem 2 is now complete. Given a sequence $\mathcal{D}$ of n disks ordered by non-increasing radius, a covering as in Lemma 1 can be obtained in $O(n)$ time. As a consequence, Theorems 1 and 2 lead to $O(n)$ time algorithms for offline covering under the same disk order assumption.

3 Online covering

In this section we prove Theorem 3. Let the *unit disk* D be a disk of unit *diameter*[5]. Denote by d_i the diameter of the ith disk D_i in the sequence $\mathcal{D}$. Denote by $|C|$ the area of a convex body C in the plane. Let $\mathbb{N} = \{0,1,2,\ldots\}$ denote the set of natural numbers, and let $\mathbb{N}^+ = \{1,2,3,\ldots\}$ denote the set of positive integers.

The unit disk D is contained in a unit square S. Each disk D_i of diameter d_i contains a square S_i of side length $s_i = d_i/\sqrt{2}$. Note that $|S|/|D| = 4/\pi$ and $|D_i|/|S_i| = \pi/2$. Therefore, using the current best bound for online covering a unit square S by squares S_i, namely $\frac{7}{4}\sqrt[3]{9} + \frac{13}{8} = 5.265\ldots$ [9], we immediately obtain a bound of

$$\eta \le \frac{4}{\pi} \cdot \frac{\pi}{2} \cdot \left(\frac{7}{4}\sqrt[3]{9} + \frac{13}{8} \right) = 10.5302\ldots.$$

By using an efficient adaptation of a method by Januszewski and Lassak [8, 9], we obtain a better bound of $\eta < 9.7633$. The idea is to use an inscribed rectangle R_i in each disk D_i, instead of an inscribed square S_i, to cover the unit square S.

We first review some basic techniques for online covering [8, 9]. Suppose we want to cover the unit square $S = [0,1]^2$ by a sequence $\mathcal{S}$ of axis-parallel squares. And suppose that each square $S_i \in \mathcal{S}$ is *normalized*: its side length has the form 2^{-r}, $r \in \mathbb{N}^+$. The method of the current bottom [8] places each square S_i as follows: First find the largest number b_i such that every point of S with y-coordinate at most b_i has been covered by some square S_j, $j < i$. The set of points of S with y-coordinate equal to b_i is called the *ith bottom*. A point of the ith bottom is called a *surface point* if no point of S with the same x-coordinate and with a larger y-coordinate has been covered by the preceding squares. Now place S_i *at the bottom*, that is, find a translation τ_i such that $\tau_i(S_i)$ contains a surface point and has the form

$$\left\{ (x,y) \mid m\,2^{-r} \le x \le (m+1)2^{-r} \text{ and } b_i \le y \le b_i + 2^{-r} \right\}, \text{ where } m \in \{0,\ldots,2^r-1\}.$$

[5]The unit disk was defined as a disk of unit *radius* in Section 2. Here we use a different definition for convenience in analysis.

Since $\tau_i(S_i)$ contains a surface point on its lower side, it does not overlap with the preceding squares that are larger or of equal size. Hence the upper half of $\tau_i(S_i)$ is not covered by the preceding squares. The lower half of $\tau_i(S_i)$ consists of the lower-left quarter and the lower-right quarter; at least one of the two quarters contains a surface point on its lower side. Apply the same argument recursively to this quarter of $\tau_i(S_i)$, and it follows that the fraction of the area of $\tau_i(S_i)$ not covered by the preceding squares is at least

$$\frac{1}{2}\left(1+\frac{1}{4}+\frac{1}{4\cdot 4}+\frac{1}{4\cdot 4\cdot 4}+\cdots\right)=\frac{1}{2}\cdot\frac{1}{1-\frac{1}{4}}=\frac{2}{3}.$$

That is, $2/3$ of the area of $\tau_i(S_i)$ is covered for the first time. Similarly observe that, above the current bottom b_i, an area of at most $1/3$ (the total area of a collection of squares, one of each side length 2^{-k}, $k=1,2,\ldots$) is covered by the squares S_j, $j<i$. Therefore, the total area of the squares preceding S_i is at most

$$\frac{3}{2}\left(b_i+\frac{1}{3}\right).$$

The unit square S becomes completely covered when b_i reaches 1. Thus a total area of $(3/2)(1+1/3)=2$ is sufficient for online covering a unit square by normalized squares. Since every square contains a normalized square of at least $1/2$ of its side length and hence at least $1/4$ of its area, it follows that $g_2(C)\le 8$ for a square C. The method of the current bottom has been extended to the method of the current bottom and top [8, 9], in which a square S_i may be placed at either the current bottom b_i or the current top t_i (which is defined analogously). Initially, $b_1=0$ and $t_1=1$. The unit square S becomes completely covered when $b_i\ge t_i$ for some i. This extended method yields the current best bound of $g_2(C)\le\frac{7}{4}\sqrt[3]{9}+\frac{13}{8}=5.265\ldots$ for a square C [9].

We now outline another way to extend the method of the current bottom. Observe that, when the sequence $\mathcal{S}$ of squares is replaced by a sequence $\mathcal{B}$ of similar rectangles with width and height of the form 2^{-r} and $u\cdot 2^{-r}$, $r\in\mathbb{N}^+$, the previous argument for the ratio $2/3$ remains valid. The total area of the rectangles preceding rectangle $B_i\in\mathcal{B}$ becomes

$$\frac{3}{2}\left(b_i+\frac{u}{3}\right).$$

Now suppose that we have another sequence $\mathcal{T}$ of similar rectangles with width and height of the form $2^{-r}/3$ and $v\cdot 2^{-r}/3$, $r\in\mathbb{N}$. To cover the unit square S from the top, place each rectangle $T_i\in\mathcal{T}$ such that $\tau_i(T_i)$ has the form

$$\left\{(x,y)\mid m\,2^{-r}/3\le x\le(m+1)2^{-r}/3 \text{ and } t_i-2^{-r}/3\le y\le t_i\right\},\ \text{where } m\in\{0,\ldots,2^r-1\}.$$

Then the total covered area below the current top becomes

$$\frac{v}{9}+\frac{v}{9}+\frac{v}{9\cdot 4}+\frac{v}{9\cdot 4\cdot 4}+\cdots=\frac{v}{9}+\frac{\frac{v}{9}}{1-\frac{1}{4}}=\frac{7v}{27}.$$

The total area of the rectangles preceding T_i becomes

$$\frac{3}{2}\left(1-t_i+\frac{7v}{27}\right).$$

We now present a method that covers the unit disk D by a sequence $\mathcal{D}$ of disks. We show that each disk $D_i\in\mathcal{D}$ of diameter d_i contains a normalized rectangle R_i of width w_i and height h_i (defined below), and use these rectangles R_i to cover the unit square S containing D. Let $1<c<2$. The exact value of c will be determined later. Consider two cases:

1. $\frac{1}{c} \cdot 2^{-k} \le d_i < 2^{-k}$, $k \in \mathbb{N}$. Then D_i contains a rectangle R_i of width $w_i = \frac{1}{2} \cdot 2^{-k}$ and height $h_i = \frac{u}{2} \cdot 2^{-k}$, where $u = \sqrt{4/c^2 - 1}$. Place R_i to cover S from the bottom.

   Define
$$f(c) = \frac{\pi}{u}.$$
   We have
$$\frac{|D_i|}{|R_i|} = \frac{\pi(d_i/2)^2}{w_i h_i} \le \frac{\pi/4}{u/4} = f(c).$$

2. $\frac{1}{2} \cdot 2^{-k} \le d_i < \frac{1}{c} \cdot 2^{-k}$, $k \in \mathbb{N}$. Then D_i contains a rectangle R_i of width $w_i = \frac{1}{3} \cdot 2^{-k}$ and height $h_i = \frac{v}{3} \cdot 2^{-k}$, where $v = \sqrt{5}/2$. Place R_i to cover S from the top.

   Define
$$g(c) = \frac{9\pi}{4c^2 v}.$$
   We have
$$\frac{|D_i|}{|R_i|} = \frac{\pi(d_i/2)^2}{w_i h_i} \le \frac{\pi/(4c^2)}{v/9} = g(c).$$

We now show that our method achieves a bound of $\eta < 9.7633$. Define

$$b(z) = \frac{3}{2}\left(z + \frac{u}{3}\right), \quad t(z) = \frac{3}{2}\left(1 - z + \frac{7v}{27}\right),$$

$$r(z,c) = \frac{b(z) \cdot f(c) + t(z) \cdot g(c)}{\pi/4}.$$

Note that $b(z) \cdot f(c)$ and $t(z) \cdot g(c)$ bound the total areas of the disks that cover the unit square S from the bottom and from the top, respectively, and that $\pi/4$ is the area of the unit disk D. Then we have

$$\eta \le \max_{0 \le z \le 1} r(z,c).$$

Now,

$$r(z,c) = \frac{4}{\pi} \cdot \frac{3}{2}\left(z + \frac{u}{3}\right) \cdot \frac{\pi}{u} + \frac{4}{\pi} \cdot \frac{3}{2}\left(1 - z + \frac{7v}{27}\right) \cdot \frac{9\pi}{4c^2 v} = \left(\frac{6}{u} - \frac{27}{2c^2 v}\right) \cdot z + 2 + \frac{27}{2c^2 v} + \frac{7}{2c^2}.$$

Let c be the solution of the following equation:

$$\frac{6}{u} - \frac{27}{2c^2 v} = 0.$$

Then $r(z,c)$ does not depend on z. A calculation shows that $c = 1.4164\ldots$ and $r(z,c) = 9.7632\ldots$. Therefore $\eta \le 9.7633$. This completes the proof of Theorem 3.

An $O(n \log n)$-time algorithm can be achieved by using a linked list to represent the "coastline" of horizontal segments bounding from above the current covered area at the bottom of the unit square, and by maintaining these segments in a priority queue. The segments bounding from below the current covered area at the top of the unit square are maintained in a similar way. The amortized cost for processing a disk is $O(\log n)$.

References

[1] P. Braß, W. Moser, and J. Pach: *Research Problems in Discrete Geometry*, Springer, New York, 2005.

[2] H. T. Croft, K. J. Falconer, and R. K. Guy: *Unsolved Problems in Geometry*, Springer, New York, 1991.

[3] Z. Füredi: Covering a triangle with homothetic copies, in *Discrete Geometry — in Honor of W. Kuperberg's 65th Birthday*, A. Bezdek, ed., Marcel Dekker, 2003, pp. 435–445.

[4] Z. Füredi: Presentation at the *AMS-IMS-SIAM Summer Research Conference "Discrete and Computational Geometry — Twenty Years Later"*, July 18–22, 2006, Snowbird, Utah, USA.

[5] J. Januszewski: On-line covering of the unit square by a sequence of convex bodies, *Demonstratio Mathematica*, **29** (1996), 155–158.

[6] J. Januszewski: Covering the unit square by squares, *Beiträge zur Algebra und Geometrie*, **43** (2002), 411–422.

[7] J. Januszewski: Translative covering a convex body by its homothetic copies, *Studia Scientiarum Mathematicarum Hungarica*, **40** (2003), 341–348.

[8] J. Januszewski and M. Lassak: On-line covering the unit cube by cubes, *Discrete & Computational Geometry*, **12** (1994), 433–438.

[9] J. Januszewski and M. Lassak: On-line covering the unit square by squares and the three-dimensional unit cube by cubes, *Demonstratio Mathematica*, **28** (1995), 143–149.

[10] J. Januszewski, M. Lassak, G. Rote, and G. Woeginger: On-line q-adic covering by the method of the n-th segment and its application to on-line covering by cubes, *Beiträge zur Algebra and Geometrie*, **37** (1996), 51–65.

[11] W. Kuperberg: On-line covering a cube by a sequence of cubes, *Discrete & Computational Geometry*, **12** (1994), 83–90.

[12] M. Lassak: On-line algorithms for the q-adic covering of the unit interval and for covering a cube by cubes, *Beiträge zur Algebra and Geometrie*, **43** (2002), 537–549.

[13] J. Moon and L. Moser: Some packing and covering theorems, *Colloquium Mathematicum*, **17** (1967), 103–110.

[14] M. Naszódi: Covering a convex body by its homothets of different sizes, Abstract presented at the *Discrete & Convex Geometry Workshop*, Rényi Institute of Mathematics, July 4–6, 2008, Budapest, Hungary.

[15] E. H. Neville: On the solution of numerical functional equations, illustrated by an account of a popular puzzle and of its solution, *Proceedings of the London Mathematical Society*, **14** (1915), 308–326.

[16] T. Rike: *Inequalities and Triangles*, `http://mathcircle.berkeley.edu/trig.pdf`, Berkeley Math Circle, December 5, 1999.

A Source code

```
#include <math.h>
#include <stdio.h>

#define RATIO    2.97
#define STEP    0.005

double find_hijk(double xi, double xj, double xk);

int main() {
    double x1, x2, x3, x4, x5;
    double x1_, x2_, x3_, x4_, x5_;
    double s1_, s2_, s3_, s4_, s5_;
    double a1, a2, a3, a4;
    double h12, h34, h345;
    double w, h, x, x_;
    double r4_, r5_;
    double start = 0.0, end = 1.0;

    printf("Testing ratio %g with step size %g ...\n", RATIO, STEP);
    for (x1 = start; x1 <= end; x1 += STEP) {
        x1_ = x1 + STEP; s1_ = x1_ * x1_;
        a1 = asin(x1);
        for (x2 = start; x2 <= x1; x2 += STEP) {
            x2_ = x2 + STEP; s2_ = s1_ + x2_ * x2_;
            a2 = asin(x2);
            h12 = cos(a1 + a2);
            for (x3 = start; x3 <= x2; x3 += STEP) {
                x3_ = x3 + STEP; s3_ = s2_ + x3_ * x3_;
                a3 = asin(x3);
                for (x4 = start; x4 <= x3; x4 += STEP) {
                    x4_ = x4 + STEP; s4_ = s3_ + x4_ * x4_;
                    a4 = asin(x4);
                    h34 = cos(a3 + a4);

                    if (h12 + h34 <= 0.0) /* condition 1 */
                        break;

                    for (x5 = start; x5 <= x4; x5 += STEP) {

                        if (x5 >= 0.6094) /* condition 2 */
                            break;

                        x5_ = x5 + STEP; s5_ = s4_ + x5_ * x5_;
                        x = x5_ * M_SQRT2;
                        w = 2.0;
                        h = h12 + h34;
                        x_ = x < h ? x : h;
                        r4_ = s4_ + (w * (h + x) + h * x_) / 2.0;

                        if (RATIO >= r4_) /* condition 3 */
                            continue;
```

```
                        h345 = find_hijk(x3, x4, x5);
                        h = h12 + h345;
                        x_ = x < h ? x : h;
                        r5_ = s5_ + (w * (h + x) + h * x_) / 2.0;

                        if (RATIO >= r5_) /* condition 4 */
                            continue;

                        /* difficult case */
                        printf("%5.3f %5.3f %5.3f %5.3f %5.3f  %5.3f %5.3f\n",
                                x1, x2, x3, x4, x5, r4_, r5_);
                        printf("    x %5.3f h12+h34 %5.3f h12+h345 %5.3f\n",
                                x, h12 + h34, h12 + h345);
                    }
                }
            }
        }
    }
    return 0;
}

double find_hijk(double xi, double xj, double xk) {
    double ak = asin(xk);
    double a, b, h;
    double alpha, beta;
    double upper = cos(ak * 2.0); /* trapezoid becomes triangle */
    double lower = -xk;    /* trapezoid becomes rectangle */

    while (upper - lower > 0.001) { /* binary search */
        h = (upper + lower) / 2.0;
        alpha = acos(h);
        beta = alpha - ak * 2.0;
        a = sin(beta);
        b = (cos(beta) - h) / 2.0;
        if (a * a + b * b <= xi * xi)
            upper = h;
        else
            lower = h;
    }
    return upper;
}
```