Computational Geometry Column 58

Adrian Dumitrescu* Minghui Jiang†

November 6, 2013

Abstract

This column is devoted to opaque sets also known as barriers. A set of curves Γ that meet every line which intersects a given convex body B is called an opaque set or barrier for B. Although the shape and length of shortest barriers for simple bodies, such as a unit equilateral triangle or a unit square are seldom disputed, no proof of optimality is known or appears to be even near in sight.

Keywords: Opaque set, opaque square problem.

1 Introduction

The problem of finding small sets that block every line passing through a unit square was first considered by Mazurkiewicz in 1916 [21]; see also [2], [13]. Let B be a convex body in the plane. Following Bagemihl [2], a set Γ is an *opaque set* or a *barrier* for B, if Γ meets all lines that intersect B. A barrier does not need to be connected; it may consist of one or more rectifiable arcs and its parts may lie anywhere in the plane, including the exterior of B; see [2, 3].

What is the length of the shortest barrier for a given convex body B? In spite of considerable efforts, the answer to this question is not known even in the simplest instances, such as when B is a square, a disk, or an equilateral triangle; see [4], [5, Problem A30], [9], [10], [11], [12, Section 8.11], [14, Problem 12]. Obviously, for any body B, a line ℓ intersects B if and only if ℓ intersects $\text{conv}(B)$, the convex hull of B; and this is the reason for the convexity restriction in the question.

A barrier blocks any line of sight across the region B or detects any ray that passes through it. Potential applications are in guarding and surveillance. One of the original motivations of the problem is mentioned by Faber et al. [10, 11]: A repairman from a telephone company, while repairing buried cable, has discovered that often the cable is not directly under the marker which is supposed to be erected above it. Assuming that the cable is straight and is always within 2 meters from the marker in a horizontal plane, what is shortest length of a trench that the repairmen has to dig such that the cable is guaranteed to be found? In the terminology of the opaque set problem, the disk of radius 2 meters centered at the marker is the convex body, the possible locations of the cable are the lines intersecting the convex body, and the trench is the barrier.

Some entertaining variants of the opaque set problem appeared in different forms [16, 19, 20]; see also [5, Problem A30]. For instance, what should a swimmer at sea do in a thick fog if he knows

*Department of Computer Science, University of Wisconsin–Milwaukee, USA. Email: dumitres@uwm.edu
†Department of Computer Science, Utah State University, Logan, USA. Email: mjiang@cc.usu.edu

that he is within a mile of a straight shoreline? Here the convex body is the disk of radius one mile centered at the start location of the swimmer, and the barrier is the route taken by the swimmer. This is almost the same problem as that for the telephone company except that the barrier here is restricted to be a single curve originating from the disk center.

The shortest barrier known for the unit square is depicted in Figure 1(right). It is conjectured to be optimal. The current best lower bound, 2, has been established by Jones [15] in 1964[1].

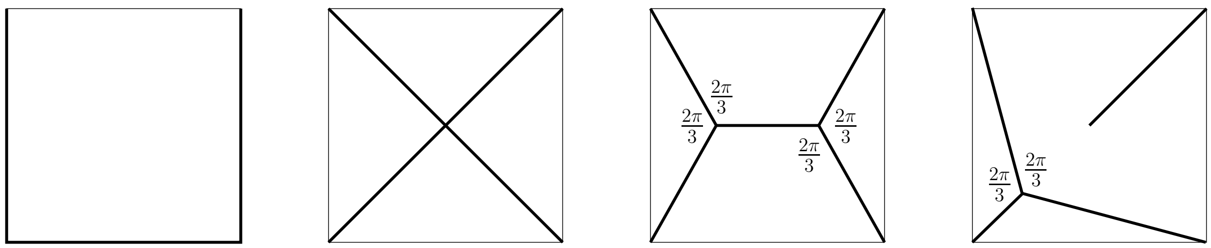

Figure 1: The first three from the left are barriers for the unit square of lengths 3, $2\sqrt{2} = 2.8284\ldots$, and $1+\sqrt{3} = 2.7320\ldots$. Right: The diagonal segment $[(1/2, 1/2), (1, 1)]$ together with three segments connecting the corners $(0, 1)$, $(0, 0)$, $(1, 0)$ to the point $(\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} - \frac{\sqrt{3}}{6})$ yield a barrier of length $\sqrt{2} + \frac{\sqrt{6}}{2} = 2.639\ldots$.

The type of curve barriers considered may vary: the most restricted are barriers made from single continuous arcs, then connected barriers, and lastly, arbitrary (possibly disconnected) barriers. For the unit square, the shortest known in these three categories have lengths 3, $1 + \sqrt{3} = 2.7320\ldots$ and $\sqrt{2} + \frac{\sqrt{6}}{2} = 2.6389\ldots$, respectively. They are depicted in Figure 1. Obviously, disconnected barriers offer the greatest freedom of design. For instance, Kawohl [18] showed that the barrier in Figure 1(right) is optimal in the class of curves with at most two components restricted to the square. For the unit disk, the shortest known barrier consists of three arcs. See also [10, 12].

Barriers can be also classified by where they can be located. For instance, it may be infeasible to construct a barrier for a specific domain outside the domain, since that part might belong to someone else. Following [7] we call such barriers constrained to the (interior and the boundary of the) domain, *interior*. For example, all four barriers for the square illustrated in Figure 1 are interior barriers. On the other hand, certain instances may prohibit barriers lying in the interior of a domain. We call a barrier constrained to the exterior and the boundary of the domain, *exterior*. As an illustration, the first barrier from the left in Figure 1 is exterior (and interior as well).

2 Preliminaries

In order to be able to speak of the *length* $\text{len}(B)$ of a barrier B, we consider rectifiable barriers. A *rectifiable curve* is a curve of finite length. A *rectifiable barrier* is the union of a countable set of *rectifiable curves*, $\Gamma = \cup_{i=1}^{\infty} \gamma_i$, where $\sum_{i=1}^{\infty} |\gamma_i| < \infty$ (or $\Gamma = \cup_{i=1}^{n} \gamma_i$ for some n). A *segment barrier*

[1] A note of caution for the non-expert about the subtlety of the problem: an arxiv submission [8] dated May 2010 claimed a first small improvement in the old lower bound of 2 for a unit square, due to Jones [15], from 1964; its proof had a fatal error, and the submission was soon after withdrawn by the authors. Further, at least two conference submissions by two other groups of authors were made in the last 3 years claiming (erroneous) improvements in the same lower bound of 2 for a unit square; both submissions were rejected at the respective conferences and the authors were notified of the errors discovered. In September 2013, yet another improvement in the lower bound of 2 for a unit square has been announced [17]. Its correctness however remains unverified, since no proof seems to be publicly available at the time of this writing.

is a barrier consisting of straight-line segments (or polygonal paths). The first step in studying the lengths of barriers is understanding segment barriers. Indeed, the shortest segment barrier is not much longer than the shortest rectifiable one:

Lemma 1. [7]. *Let B be a rectifiable barrier for a convex body C in the plane. Then, for any $\varepsilon > 0$, there exists a segment barrier B_ε for C, consisting of a countable set of straight-line segments, such that* $\mathrm{len}(B_\varepsilon) \le (1+\varepsilon)\,\mathrm{len}(B)$.

A key fact in establishing a constant approximation ratio is a lower bound on the length of a barrier offered by Cauchy's integral formula. Its proof is folklore; see for instance [7, 11, 15]. Denote by $\mathrm{per}(B)$ the perimeter of a convex body B in the plane.

Lemma 2. *Let B be a convex body in the plane and let Γ be a barrier for B. Then the length of Γ is at least $\frac{1}{2} \cdot \mathrm{per}(B)$, where $\mathrm{per}(B)$ denotes the perimeter of B.*

The above lower bound is tight for a (thin rectangle that degenerates to a) segment s. In this case, a barrier of length s, i.e., the segment itself, suffices and is also necessary.

3 In search of exact algorithms and good approximations

In the late 1980s, Akman [1] soon followed by Dublish [6] had reported algorithms for computing a minimum interior-restricted barrier of a given convex polygon. Both algorithms were shown to be incorrect however by Shermer [23] in 1991, who proposed a new exact algorithm instead. Shermer conjectured that a shortest interior-restricted barrier (he calls it an "opaque forest") of a convex polygon can be generated by an instance of the following procedure:

(a) Find a triangulation T of P.

(b) Remove zero or more diagonals of T, so that at most one nontriangular interior region U is formed. Let the edges of U's Steiner tree be in the opaque forest.

(c) For all triangles of T (other than U, if U is a triangle), let the height of the triangle (using the edge topologically closest to U as the base) be in the opaque forest.

Recently, Provan et al. [22] refuted Shermer's conjecture with a convex polygon as simple as a rhombus; see Figure 2. Specifically, their example shows that Shermer's procedure does not necessarily compute the shortest *interior-restricted* barrier or the shortest *unrestricted* barrier. As of today, no exact algorithm for computing a shortest (interior-restricted or unrestricted) barrier is known.

Figure 2: Two interior-restricted barriers for a rhombus with diagonals of lengths 2 and 16. Left: the barrier computed by Shermer's procedure has length 17.124 . . .. Right: the barrier found by Provan et al. has length 17.035 . . ..

Even though we have so little control on the shape or length of optimal barriers, barriers whose lengths are relatively close to optimal can be computed efficiently for any convex polygon. Various approximation algorithms with a small constant ratio (below 1.6) have been obtained recently [7] and are listed next. Let P be a given convex polygon with n vertices.

(i) A (possibly disconnected) barrier for P, whose length is at most $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$ times the optimal, can be computed in $O(n)$ time.

(ii) A connected polygonal barrier whose length is at most 1.5716 times the optimal can be computed in $O(n)$ time.

(iii) A single-arc polygonal barrier whose length is at most $\frac{\pi+5}{\pi+2} = 1.5834\ldots$ times the optimal can be computed in $O(n)$ time.

(iv) An optimal interior single-arc barrier can be computed in $O(n^2)$ time.

(v) An interior connected barrier whose length is at most $(1+\varepsilon)$ times the optimal can be found in polynomial time.

To avoid any confusion we emphasize that the approximation ratios are for each barrier class, that is, the length of the barrier computed is compared to the optimal length in the corresponding class. For example, the connected barrier computed by the approximation algorithm with ratio 1.5716 is *not* necessarily shorter than the (possibly disconnected) barrier computed by the approximation algorithm with the larger ratio $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$.

For illustration we give a short outline of the approximation algorithm from [7] for arbitrary (possibly disconnected) barriers. First compute a minimum-perimeter rectangle R containing P; refer to Figure 3. Let P_i, $i = 1, 2, 3, 4$ be the four polygonal paths on P's boundary, connecting

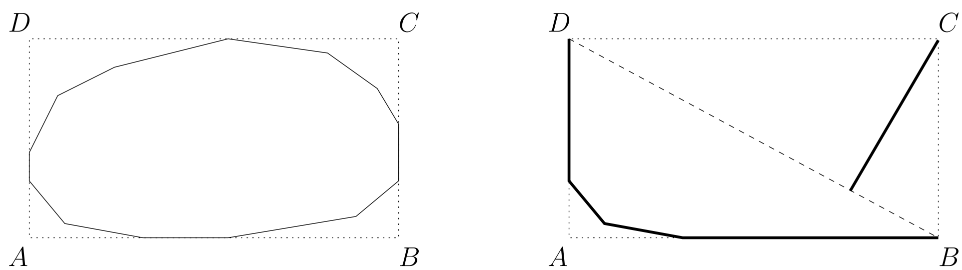

Figure 3: The approximation algorithm for arbitrary barriers.

the four (possibly degenerate) segments on the boundary of P contained in the left, bottom, right and top side of R. Consider four barriers for P, denoted Γ_i, for $i = 1, 2, 3, 4$. Γ_i consists of the polygonal path P_i extended at both ends on the corresponding rectangle sides, plus the height from the opposite rectangle vertex in the complementary right-angled triangle; see Figure 3 (right). The algorithm returns the shortest of these four barriers. It is then shown that its length is at most $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$ times $\text{per}(P)/2$; it follows from Lemma 2 that the approximation ratio of the algorithm is $\frac{1}{2} + \frac{2+\sqrt{2}}{\pi} = 1.5867\ldots$, as required.

4 Open problems

The topic of opaque barriers remains as fascinating as hundred years ago when it was first introduced. We conclude with a few open problems:

1. Prove (or disprove) that the shortest barrier for any convex polygon with n vertices consists of $O(n)$ segments, perhaps at most $n + c$, for a suitable constant c.

2. Can one give a characterization of the class of convex polygons whose optimal barriers are (i) interior? (ii) exterior? (iii) neither interior nor exterior?
3. Is there a polynomial-time algorithm for computing a shortest barrier for a given convex polygon with n vertices? Is there one for computing a shortest interior (resp., exterior) barrier?
4. Can the approximation ratios from [7] for arbitrary, or connected, or single-arc barriers be improved?
5. Can a lower bound of $2+\delta$ on the length of a shortest barrier for the unit square be established, for some constant $\delta > 0$?
6. Prove (or disprove) that an optimal barrier for the square is interior. Prove (or disprove) that an optimal exterior barrier for the unit square has length 3.

References

[1] V. Akman, An algorithm for determining an opaque minimal forest of a convex polygon, *Information Processing Letters*, **24** (1987), 193–198.

[2] F. Bagemihl, Some opaque subsets of a square, *Michigan Math. J.*, **6** (1959), 99–103.

[3] K. A. Brakke, The opaque cube problem, *American Mathematical Monthly*, **99(9)** (1992), 866–871.

[4] H. T. Croft, Curves intersecting certain sets of great-circles on the sphere, *J. London Math. Soc. (2)* **1** (1969), 461–469.

[5] H. T. Croft, K. J. Falconer, and R. K. Guy, *Unsolved Problems in Geometry*, Springer, New York, 1991.

[6] P. Dublish, An $O(n^3)$ algorithm for finding the minimal opaque forest of a convex polygon, *Information Processing Letters*, **29(5)** (1988), 275–276.

[7] A. Dumitrescu, M. Jiang, and J. Pach, Opaque sets, *Algorithmica*, to appear. Online first, December 2012; DOI 10.1007/s00453-012-9735-2.

[8] A. Dumitrescu and J. Pach, Opaque sets, manuscript, May 12, 2010, `arXiv:1005.2218v1`.

[9] H. G. Eggleston, The maximal in-radius of the convex cover of a plane connected set of given length, *Proc. London Math. Soc. (3)*, **45** (1982), 456–478.

[10] V. Faber and J. Mycielski, The shortest curve that meets all the lines that meet a convex body, *American Mathematical Monthly*, **93** (1986), 796–801.

[11] V. Faber, J. Mycielski and P. Pedersen, On the shortest curve which meets all the lines which meet a circle, *Ann. Polon. Math.*, **44** (1984), 249–266.

[12] S. R. Finch, *Mathematical Constants*, Cambridge University Press, 2003.

[13] H. M. S. Gupta and N. C. B. Mazumdar, A note on certain plane sets of points, *Bull. Calcutta Math. Soc.*, **47** (1955), 199–201.

[14] R. Honsberger, *Mathematical Morsels*, Dolciani Mathematical Expositions, No. 3, The Mathematical Association of America, 1978.

[15] R. E. D. Jones, Opaque sets of degree α, *American Mathematical Monthly*, **71** (1964), 535–537.

[16] H. Joris, Le chasseur perdu dans le foret: une problème de géométrie plane, *Elemente der Mathematik*, **35** (1980), 1–14.

[17] A. Kawamura, S. Moriyama, and Y. Otachi, On shortest barriers, communication at the *16th Japan Conference on Discrete and Computational Geometry and Graphs* (JCDCG2 2013), September 17–19, 2013, Tokyo, Japan.

[18] B. Kawohl, Some nonconvex shape optimization problems, in *Optimal Shape Design* (A. Cellina and A. Ornelas, editors), vol. 1740/2000 of Lecture Notes in Mathematics, Springer, 2000.

[19] R. Klötzler, Universale Rettungskurven I, *Zeitschrifte für Analysis und ihre Anwendungen*, **5** (1986), 27–38.

[20] R. Klötzler and S. Pickenhain, Universale Rettungskurven II, *Zeitschrifte für Analysis und ihre Anwendungen*, **6** (1987), 363–369.

[21] S. Mazurkiewicz, Sur un ensemble fermé, punctiforme, qui rencontre toute droite passant par un certain domaine (Polish, French summary), *Prace Mat.-Fiz.*, **27** (1916), 11–16.

[22] J. S. Provan, M. Brazil, D. A. Thomas and J. F. Weng, Minimum opaque covers for polygonal regions, manuscript, October 2012, `arXiv:1210.8139v1`.

[23] T. Shermer, A counterexample to the algorithms for determining opaque minimal forests, *Information Processing Letters*, **40** (1991), 41–42.