Computational Geometry Column 56

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Abstract

This column is devoted to maximum (respectively, maximum weight) independent set problems in geometric intersection graphs. We illustrate with one example in each class:

(I) The following question was asked by T. Rado in 1928: What is the largest number c such that, for any finite set $\mathcal{F}$ of axis-parallel squares in the plane, there exists an independent subset $\mathcal{I} \subseteq \mathcal{F}$ of pairwise disjoint squares with total area at least c times the union area of the squares in $\mathcal{F}$?

(II) The following question was asked by Erdős in 1983: What is the largest number $H = H(n)$ with the property that every set of n non-overlapping unit disks in the plane has an independent subset with at least H members?

Keywords: Geometric intersection graph, maximum independent set, maximum area independent set, approximation algorithm.

1 Preliminaries

Maximum Independent Set (MIS) is the problem of finding an independent set of maximum cardinality in a given undirected graph. In the weighted version, Maximum Weight Independent Set (MWIS), each vertex of the graph has a non-negative weight, and the problem is to find an independent set of maximum total weight. MWIS is often studied in geometric settings, where the input graph $G = (V, E)$ is the intersection graph of a set V of geometric objects. The *intersection graph* $G(\mathcal{S})$ of a set $\mathcal{S}$ of objects has a vertex representing each object in $\mathcal{S}$ and an edge between two vertices if and only if the corresponding objects intersect in their interiors [14]. It is also possible to define the intersection graph for pairwise interior-disjoint objects with edges between objects in contact. For MWIS in geometric intersection graphs, a common choice for the weight is the volume (or area) of the geometric object.

While MIS and MWIS lead to hard computational problems even in the simplest settings [5, 6, 11], here we focus on quantitative bounds and the combinatorial aspects of these problems which are interesting in their own right.

Definitions and notations. The term *convex body* refers to a compact convex set with nonempty interior. For a convex body S in $\mathbb{R}^d$, denote by $|S|$ the Lebesgue measure of S, i.e., the length when $d = 1$, the area when $d = 2$, or the volume when $d \geq 3$. For a finite set $\mathcal{F}$ of convex bodies in

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$\mathbb{R}^d$, denote by $|\mathcal{F}| = |\cup_{S\in\mathcal{F}} S|$ the Lebesgue measure of the union of the convex bodies in $\mathcal{F}$; when $d = 2$, we call $|\mathcal{F}|$ the *union area* of $\mathcal{F}$. For a convex body S in $\mathbb{R}^d$, define

$$F(S) = \inf_{\mathcal{F}} \sup_{\mathcal{I}} \frac{|\mathcal{I}|}{|\mathcal{F}|},$$

where $\mathcal{F}$ ranges over all finite sets of convex bodies in $\mathbb{R}^d$ that are homothetic to S, and $\mathcal{I}$ ranges over all independent subsets of $\mathcal{F}$. Also define

$$f(S) = \inf_{\mathcal{F}_1} \sup_{\mathcal{I}} \frac{|\mathcal{I}|}{|\mathcal{F}_1|},$$

where $\mathcal{F}_1$ ranges over all finite sets of convex bodies in $\mathbb{R}^d$ that are homothetic and *congruent* to S, and $\mathcal{I}$ ranges over all independent subsets of $\mathcal{F}_1$.

For instance, R. Rado [19, Theorem 10 iii] showed that for a disk S, we have $f(S) \geq \frac{\pi}{8\sqrt{3}} > 1/4.4107$. This remains the current best lower bound today. For another example, if S is a square, we have $f(S) = 1/4$ [16, 23, 25], however $F(S) < 1/4$ [1]. In general, for an arbitrary convex body S in the plane, we have $f(S) \geq 1/6$ [3] and $f(S) \leq 1/4$ [19]. The lower bound $f(S) \geq 1/6$ cannot be improved, as R. Rado showed that $f(S) = 1/6$ for any triangle S [19, Theorem 10]; likewise, the upper bound $f(S) \leq 1/4$ cannot be improved either, since $f(S) = 1/4$ for a square S. The current record upper and lower bounds for Rado's problem in the plane (still with gaps in most cases) are displayed in Table 1. Various algorithmic implementations realizing these lower bounds are obtained in [2, 3].

Convex body S	**Lower bound**	**Upper bound**
Square	$f(S) \geq 1/4$ [16, 23, 25]	$f(S) \leq 1/4$ [22]
Square	$F(S) > 1/8.4797$ [3]	$F(S) \leq 1/4 - 1/384$ [3]
Disk	$f(S) > 1/4.4107$ [19]	$f(S) \leq 1/4$ [19]
Disk	$F(S) > 1/8.3539$ [3]	$F(S) \leq 1/4$ [19]
Centrally symmetric	$f(S) > 1/4.4810$ [3]	$f(S) \leq 1/4$ [19]
Centrally symmetric	$F(S) > 1/8.5699$ [3]	$F(S) \leq 1/4$ [19]

Table 1: Lower and upper bounds for Rado's problem in the plane. Note that one does not expect a tight bound in the last row, e.g., because $F(S) < 1/4$ [1] for a square S, while it is conjectured that $F(S) \geq 1/4$ [2] for a disk S.

2 Rado's problem and a lattice technique

Rado's problem on selecting disjoint squares is a famous unsolved problem in geometry [7, Problem D6]: T. Rado [22] conjectured in 1928 that if $\mathcal{F}$ is a finite set of axis-parallel squares in the plane, then there exists an independent subset $\mathcal{I} \subseteq \mathcal{F}$ of pairwise disjoint squares, such that $\mathcal{I}$ covers at least 1/4 of the area covered by $\mathcal{F}$. He also observed that a greedy algorithm, which repeatedly selects the largest square disjoint from those previously selected, finds an independent subset $\mathcal{I}$ of disjoint squares with total area at least 1/9 of the area of the union of all squares in $\mathcal{F}$. This lower bound has been only slightly improved over the last 80 years: by R. Rado [19] to 1/8.75, by Zalgaller [25] to 1/8.6, and by Bereg et al. [3] to 1/8.4797. On the other hand, an upper bound of 1/4 for the area ratio is obvious: take four unit squares sharing a common vertex, then only one of them may be selected. Of course, one can generate arbitrarily large examples with the same ratio using this pattern.

For congruent squares, the conjectured ratio 1/4 was confirmed by Norlander [16], Sokolin [23], and Zalgaller [25]. In 1973 (after 45 years), Ajtai [1] came up with a surprising example disproving T. Rado's conjecture. However, since then (i.e., in 40 years), no one has been able to reformulate this conjecture, i.e., replace 1/4 with another value suspected to be tight. The current best upper bound $F(S) \leq 1/4 - 1/384$ [3] is obtained by analyzing the construction[1] in Fig. 1.

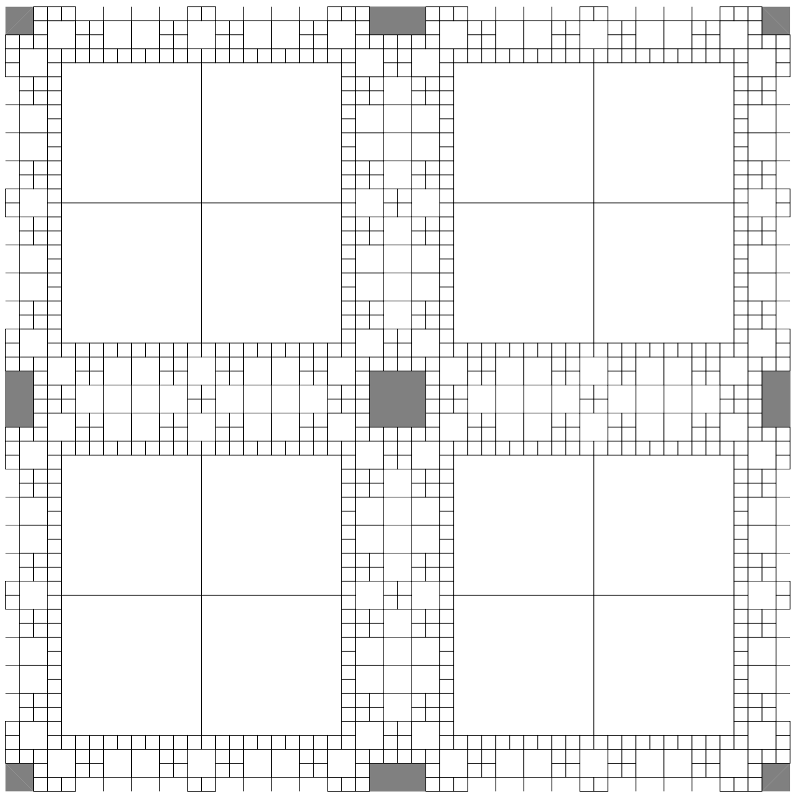

Figure 1: Tiling of the plane with blocks composed of 4 large squares of side 10 bordered by 4 rotated copies of a system of smaller squares of side 1 and 2 (some of these are shared between adjacent blocks). The shaded rectangles in the figure represent holes in the tiling, they are not part of the square system.

The analogous question for other shapes and many similar problems have been considered by R. Rado in a more general setting for various classes of convex bodies, in his three papers entitled "Some covering theorems" [19, 20, 21].

In higher dimensions the current bounds are even weaker. Given a set $\mathcal{F}$ of n axis-parallel hypercubes in $\mathbb{R}^d$, $d \geq 2$, an independent set $\mathcal{I} \subseteq \mathcal{F}$ such that $|\mathcal{I}|/|\mathcal{F}| \geq 1/3^d$ can be easily found by the greedy algorithm. Only recently a slightly better bound has been established [3]: an independent set $\mathcal{I} \subseteq \mathcal{F}$ such that $|\mathcal{I}|/|\mathcal{F}| \geq 1/\lambda_d$ can be computed in $O(dn^2)$ time, where λ_d is the unique solution in $[(5/2)^d, 3^d]$ to the equation

$$3^d - (\lambda^{1/d} - 2)^d/2 = \lambda.$$

[1]The figure is reproduced from [3].

Sadly however, one can show that $\lim_{d\to\infty}(\lambda_d)^{1/d} = 3$, so this bound is not too far from the previous trivial one.

We review the proof for the bound $f(S) = 1/4$ when S is a square. As mentioned earlier, the upper bound $f(S) \leq 1/4$ is easily attained by four squares sharing a common vertex. Rado [19] proved the matching lower bound $f(S) \geq 1/4$ using an elegant lattice technique pioneered by Minkowski [15]:

Refer[2] to Figure 2. Let $\mathcal{F}$ be a finite family of axis-parallel unit squares. Consider a square lattice Λ where each cell is a square of side length 2. Fix an arbitrary cell of the lattice, say σ. Translate all lattice cells partially or totally covered by the union of the squares in $\mathcal{F}$ to σ. Put $k := \lceil |\mathcal{F}|/4 \rceil$. Since the union area of $\mathcal{F}$ is $|\mathcal{F}|$ and the area of σ is 4, there exists a point p in σ covered at least k times. This means that there exist k distinct points p_i in k distinct cells σ_i of Λ respectively, $1 \leq i \leq k$, with the same relative offset as p in σ, that are covered by the union. For each point p_i, let S_i be an arbitrary square in $\mathcal{F}$ that covers p_i. Then the k squares S_i, $1 \leq i \leq k$, are pairwise disjoint, with total area at least $k \geq |\mathcal{F}|/4$, as required.

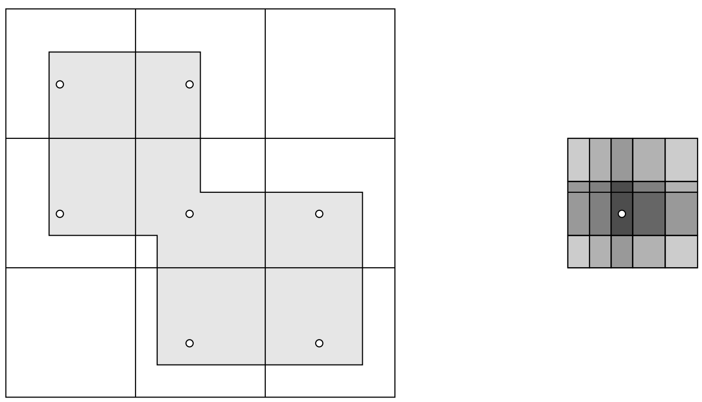

Figure 2: The lower bound $f(S) \geq 1/4$ by a lattice technique. Left: the shaded area is the union of a finite family $\mathcal{F}$ of unit squares. Right: translate all lattice cells of Λ that are covered by the union to overlap with a fixed cell σ.

The same technique was also used by Rado [19] to prove the lower bound $f(S) \geq \frac{\pi}{8\sqrt{3}} = 1/4.41\ldots$ when S is a disk, where the square lattice is replaced by a triangular lattice, corresponding to the most efficient triangular lattice packing of congruent disks in the plane.

For any finite family $\mathcal{F}$ of convex bodies, the *packing number* $\nu(\mathcal{F})$ is the maximum cardinality of a set of pairwise-disjoint bodies in $\mathcal{F}$, and the *transversal number* $\tau(\mathcal{F})$ is the minimum cardinality of a set of points that intersects every body in $\mathcal{F}$ (i.e., each body is "pierced" by at least one point in the set). The lattice technique of Rado was also used in [9] to prove the following general bounds:

Lemma 1 (Lemma 4 in [9]). *Let $\mathcal{F}$ be a finite family of translates of a centrally symmetric convex body S in $\mathbb{R}^d$. If there is a lattice packing in $\mathbb{R}^d$ with translates of S whose packing density is δ, $\delta \leq 1$, then $\nu(\mathcal{F}) \geq \frac{\delta}{2^d} \cdot |\mathcal{F}|/|S|$.*

Lemma 2 (Lemma 3 in [9]). *Let $\mathcal{F}$ be a finite family of translates of a centrally symmetric convex body S in $\mathbb{R}^d$. If there is a lattice covering in $\mathbb{R}^d$ with translates of S whose covering density is θ, $\theta \geq 1$, then $\tau(\mathcal{F}) \leq \theta \cdot |\mathcal{F}|/|S|$.*

[2]The figure is reproduced from [10].

In particular, the lower bounds $f(S) \geq 1/4$ when S is a square and $f(S) \geq \frac{\pi}{8\sqrt{3}}$ when S is a disk, follow from Lemma 1 using a square lattice packing of squares with density 1 and a triangular lattice packing of disks with density $\frac{\pi}{2\sqrt{3}}$, respectively.

Given a finite family of sets, Hall's classical marriage theorem [12, 13] provides a necessary and sufficient condition for the existence of a system of distinct representatives for the sets in the family. In the geometric setting, given a finite family of objects in the Euclidean space, a sufficient condition was given in [10] for the existence of a system of representatives that are not only distinct but also *distant* from each other. The core result in [10] is the following theorem in which the Lebesgue measure and the lattice technique play a crucial role:

Theorem 1 (Theorem 1 in [10])**.** *Let $\mathcal{F}$ be a family of n axis-parallel cubes in $\mathbb{R}^d$, and let t be a positive integer. Suppose that there exists $x > 0$ such that the following holds: for any k, $1 \leq k \leq n$, and for any subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size k, the volume of the union of the k cubes in $\mathcal{F}'$ is at least $2^d t k x^d$. Then one can choose tn points, with t points in each of the n cubes in $\mathcal{F}$, such that all pairwise distances among these points are at least x.*

3 A problem of Erdős and its extension

The following question was asked by Erdős in 1983 [17]: What is the largest number $H = H(n)$ with the property that every set of n non-overlapping unit disks in the plane has an independent subset with at least H members? The question can be viewed as a special case of Rado's problem for non-overlapping unit disks. Since for non-overlapping disks (unit or not) the intersection graph is 4-colorable [18], where each color class forms an independent set, the largest color class has at least $n/4$ members; therefore $H(n) \geq \lceil n/4 \rceil$. Csizmadia [8] improved this lower bound to $\lceil 9n/35 \rceil$, and Swanepoel [24] further raised it to $\lceil 8n/31 \rceil$. On the other hand, the construction of Chung, Graham and Pach [17] shown in Fig. 3 yields $H(n) \leq \lceil 6n/19 \rceil$. Indeed, every independent set I has at most 6 elements: one can assume by symmetry that $p \notin I$, and each shaded triangle contains at most one element in I. A refined and more involved upper bound construction due

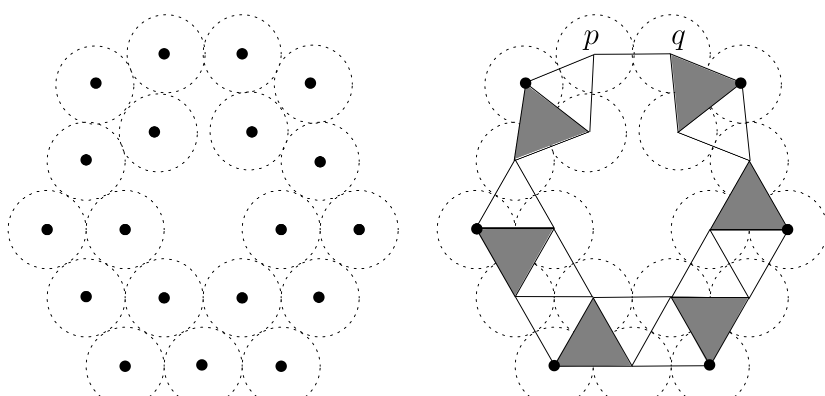

Figure 3: A construction of Chung, Graham and Pach that gives $H(n) \leq \lceil 6n/19 \rceil$.

to Pach and Tóth [17] yields $H(n) \leq \lceil 5n/16 \rceil$. So the current best bounds [4, pp. 227] are $\lceil 8n/31 \rceil \leq H(n) \leq \lceil 5n/16 \rceil \approx 0.3125n$.

The question of Erdős for non-overlapping disks can be extended to disks of arbitrary radii in the spirit of Rado; see [2]. Let $\mathcal{S}$ be a set of non-overlapping closed disks in the plane. Then there is a subset $\mathcal{I} \subseteq \mathcal{S}$ of disjoint disks whose total area is at least $|\mathcal{S}|/4$, and there are examples where no such subset has area more than $0.3028|\mathcal{S}|$. The lower bound follows from the planarity of the intersection graph [18]. The current best upper bound, 0.3028 is achieved by the construction[3] in Fig. 4.

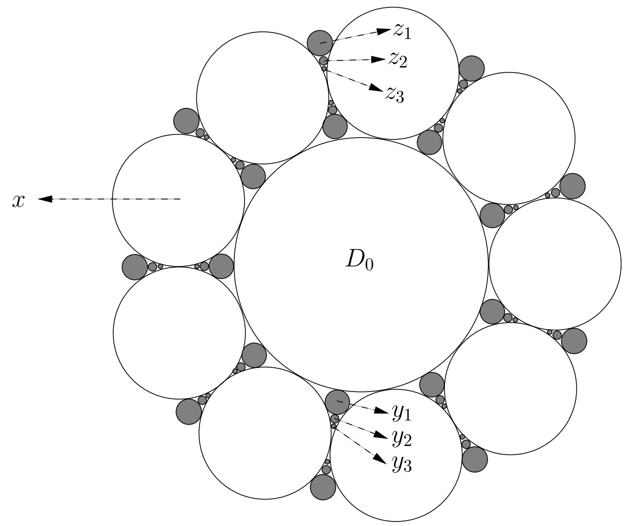

Figure 4: A unit disk, nine disks of radius $x \approx 0.5198$, nine infinite sequences of smaller disks of radii $y_1 > y_2 > \ldots$, where $y_1 \approx 0.0967$, $y_2 \approx 0.0365$, etc., and nine infinite sequences of smaller disks of radii $z_1 > z_2 > \ldots$, where $z_1 \approx 0.0956$, $z_2 \approx 0.0363$, etc.

4 Open problems

We conclude with a few open problems:

1. [3] What is the largest number c such that, for any finite set $\mathcal{F}$ of (not necessary congruent) closed axis-parallel squares in the plane, there exists an independent subset $\mathcal{I} \subseteq \mathcal{F}$ of pairwise-disjoint squares such that $|\mathcal{I}|/|\mathcal{F}| \geq c$? It seems hard to believe that $c = \frac{1}{4} - \frac{1}{384}$. The current best lower bound, from [3], is quite far, namely $|\mathcal{I}|/|\mathcal{F}| \geq 1/8.4797\ldots$.

2. [3] What is the largest number c_d such that, for any finite set $\mathcal{F}$ of (not necessary congruent) closed axis-parallel hypercubes in $\mathbb{R}^d$, there exists an independent subset $\mathcal{I} \subseteq \mathcal{F}$ of pairwise-disjoint hypercubes such that $|\mathcal{I}|/|\mathcal{F}| \geq c_d$? Can at least $\lim_{d\to\infty}(c_d)^{1/d} < 3$ be established?

3. [3] What can be said about Rado's problem for cubes in $\mathbb{R}^3$? What bounds can be derived in this case?

4. [2] For any set $\mathcal{F}$ of (not necessary congruent) closed disks in the plane, does there exist a subset $\mathcal{I}$ of pairwise-disjoint disks such that $|\mathcal{I}|/|\mathcal{F}| \geq \frac{1}{4}$? Can this statement at least

[3]The figure is reproduced from [2].

be established for congruent disks? Recall that the current best bound for congruent disks, from [19], is more than 60 years old: $f(S) \geq \frac{\pi}{8\sqrt{3}} > 1/4.4107$.

5. [2] What is the largest number c such that, for any finite set $\mathcal{F}$ of non-overlapping closed disks in the plane, there exists an independent subset $\mathcal{I} \subseteq \mathcal{F}$ of pairwise disjoint disks with total area at least c times the union area of the disks in $\mathcal{F}$? Recall that $1/4 \leq c < 0.3028$.

6. [17] What is the largest number $H = H(n)$ with the property that every set of n non-overlapping unit disks in the plane has an independent subset with at least H members? Recall that $\lceil 8n/31 \rceil \leq H(n) \leq \lceil 5n/16 \rceil$.

7. [9] Let $\mathcal{F}$ be a family of translates of a centrally symmetric convex body S in the plane. Does the inequality $\tau(\mathcal{F}) \leq |\mathcal{F}|/|S|$ hold?

8. [9] Let $\mathcal{F}$ be a family of translates of a centrally symmetric convex body S in the plane. Does the inequality $\nu(\mathcal{F}) \geq \frac{1}{4} \cdot |\mathcal{F}|/|S|$ hold?

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