Computational Geometry Column 54

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Abstract

This column is devoted to non-crossing configurations in the plane realized with straight line segments connecting pairs of points from a finite ground set. Graph classes of interest realized in this way include matchings, spanning trees, spanning cycles, and triangulations. We review some problems and results in this area. At the end we list some open problems.

Keywords: Geometric graph, non-crossing property, perfect matching, spanning tree, Hamiltonian cycle, triangulation.

1 Non-crossing configurations: setup

We review some problems regarding geometric graphs that are non-crossing. A finite set of points in the plane is said to be in *general position* if no three points are collinear. A *geometric graph* G is a pair (V, E) where V is a finite set of points (vertices) in the plane in general position, and E is a set of line segments (edges) between pairs of points in V. A geometric graph G is *non-crossing* if its edges have pairwise disjoint (relative) interiors. Note that the abstract graph corresponding to a non-crossing geometric graph is always planar, but some geometric realizations of a planar graph may have crossing edges.

2 Extremal combinatorics of non-crossing configurations

Given a set of n points in the plane in general position, every maximal non-crossing configuration is a *triangulation* in which the bounded faces are triangles and the outer face is the convex hull of the point set. Evidently, a non-crossing graph on n points has at most $3n - 6$ edges. Every *complete* geometric graph on n vertices contains a triangulation, spanning cycle [23] (also known as *polygonization*), and a perfect matching (if n is even), each of them non-crossing. However, if a geometric graph has few edges, one may only find pairwise crossing edges. This leads to a Turán-type question raised by Erdős, by Avital and Hanani [7], and by Perles and Kupitz [29]: what is the minimum number of edges in an n-vertex geometric graph that guarantees the existence of a large non-crossing subgraph (of a certain type)?

Let $e_k(n)$ be the smallest number such that any geometric graph with n vertices and $m \geq e_k(n)$ edges contains a non-crossing matching with k edges, where $k \leq n/2$ (Fig. 1, left). It is known that

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$e_k(n) = \Omega(kn)$ [29] and $e_k(n) = O(k^2 n)$ [42], and it is believed that $e_k(n) = \Theta(kn)$. A geometric graph with n vertices and $m = \Omega(k^2 n)$ edges already contains a non-crossing path with k edges [42], but it is conjectured that $m = \Omega(kn)$ edges suffice. This has been confirmed by Perles in the special case that all n vertices are in convex positions [8, 26]. To find a non-crossing Hamiltonian path in an n-vertex geometric graph, $\binom{n}{2} - \sqrt{n/2}$ edges are enough [10], but perhaps already $\binom{n}{2} - n/2$ edges suffice. On the other hand, a non-crossing Hamiltonian cycle is guaranteed by only $\binom{n}{2}$ edges, e.g., if the vertices are in convex position.

Better bounds are known for some special cases of dense geometric graphs. Consider the complete k-partite geometric graph G obtained by partitioning n points in the plane into $k = O(1)$ color classes, and connecting every two points of different colors. If the points are in convex position, then G contains a non-crossing path of at least $n - \ell$ edges, where ℓ is the size of a largest color class [20, 31]. But the best possible bound is not known even if the vertices are in convex position and all color classes have the same cardinality [1, 30] (Fig. 1, middle). In a Ramsey-type variant, the edges of a complete geometric graph on n vertices are 2-colored. Then one of the color classes always contains a non-crossing spanning tree, a non-crossing matching of $\lfloor n/3 \rfloor$ edges [25], and non-crossing cycles of size $3, 4, \ldots, \lfloor \sqrt{n/2} \rfloor$ [26]. These numbers cannot be improved. One of the color classes also contains a non-crossing path of $\Omega(n^{2/3})$ edges, which is better than the $\Omega(\sqrt{n})$ bound for dense graphs [42], but falls short of the conjectured $\Omega(n)$.

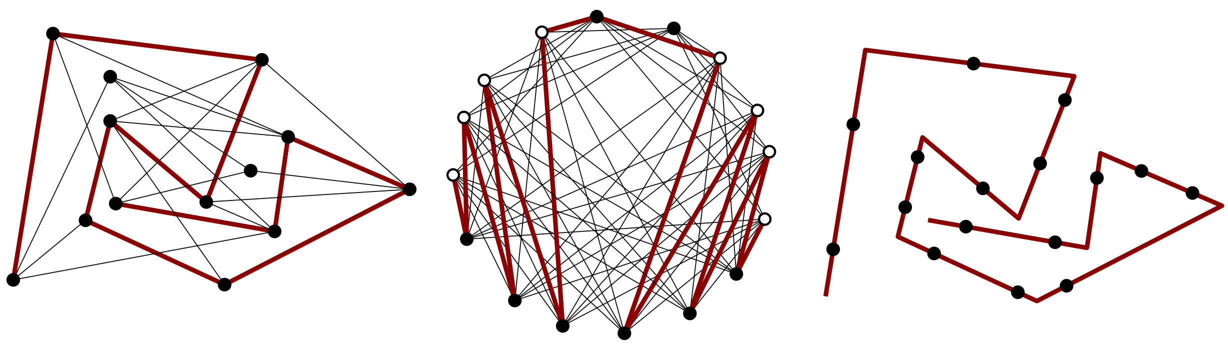

Figure 1: Left: a geometric graph on 13 vertices and a non-crossing path with 10 edges. Middle: a geometric complete bipartite graph on 16 vertices in convex position, and a non-crossing path with 14 edges. Right: a set of 16 points in general position, and a non-crossing covering path with 10 segments.

As an application of the upper bound $e_k(n) = O(k^2 n)$ [42], Araujo et al. [4] have shown that the edge set of every complete geometric graph on n vertices can be partitioned into $O(n^{3/2})$ non-crossing matchings (i.e., each composed of pairwise disjoint segments). It is an exciting open problem to decide whether a linear number of such matchings suffice, as in the case of convex complete geometric graphs. It should be noted that even the conjectured bound $e_k(n) = O(kn)$ would only give a decomposition into $O(n \log n)$ non-crossing matchings [4], thus a new approach is probably needed.

3 Non-crossing configurations: finding a good one

In most applications of geometric graphs, edge crossings are undesirable. Many structures studied in computational geometry, such as minimum spanning trees, shortest traveling salesman tours or Delaunay triangulations, are non-crossing either by definition or by their nature. For structures of minimum length, the non-crossing property often comes for free from the triangle inequality. For example, the shortest matching and the shortest spanning tree on n given points in the plane are automatically non-crossing and can be computed in polynomial time [18, 32]. For maximization problems, however, the non-crossing property is in direct conflict with the objective to maximize the Euclidean length. This interplay makes these problems attractive but also harder to deal with.

Maximization problems for geometric network design under the non-crossing constraint were first studied by Alon, Rajagopalan and Suri [6]. Given a set of n points in the plane in general position, the most natural problems are computing a longest non-crossing spanning tree, matching, Hamiltonian path or cycle, or triangulation.

Graph class	**Approximation ratio**
Perfect matching	$2/\pi \approx 0.6366$ [6]
Spanning tree	0.502 [16]
Hamiltonian path	$2/(\pi + 1) \approx 0.4829$ [16]
Hamiltonian cycle	—

Table 1: Classes of non-crossing geometric graphs, current best approximation ratios.

It is suspected (but not known) that these problems are NP-hard. On the other hand, Alon et al. [6] gave constant-ratio approximations for the first three problems above: specifically, $1/2$ for spanning trees, $2/\pi$ for matchings, and $1/\pi$ for paths. For instance the longest star (out of the n possible starts centered at one of the n points) is obviously non-crossing and yields a $\frac{1}{2}$-approximation for the longest spanning tree. Some improvements have been obtained in [16]. The current best results are summarized in Table 1. Hamiltonian cycles appear to be the hardest to approximate and no constant-ratio approximation algorithm is known (see however [16] for some special cases). Alon et al. [6] mentioned that their techniques can be applied to achieve constant factor approximations for the longest triangulation and the longest bounded-degree spanning tree.

4 Non-crossing configurations: how many can there be?

Determining the maximum number of non-crossing geometric graphs on n points in the plane is a fundamental question in combinatorial geometry. Following a common notation, we denote by $\texttt{pg}(P)$ the number of non-crossing geometric graphs that can be embedded over a point set $P \subset \mathbb{R}^2$, and by $\texttt{pg}(n) = \max_{|P|=n} \texttt{pg}(P)$ the *maximum number* of non-crossing graphs an n-element point set can admit. According to the—by now—classic result due to Ajtai et al. [3], $\texttt{pg}(n) = O(c^n)$ for some constant $c > 0$. Analogously, the maximum number of triangulations, perfect matchings, spanning trees, and spanning cycles (i.e., Hamiltonian cycles) over an n-element point set are denoted by $\texttt{tr}(n), \texttt{pm}(n), \texttt{st}(n)$ and $\texttt{sc}(n)$, respectively.

Various upper and lower bounds for common non-crossing graph classes have been obtained in [2, 9, 13, 15, 21, 24, 33, 34, 35, 36, 37, 38, 39, 40]. The current record upper and lower bounds are displayed in Table 2. Comprehensive lists of up-to-date bounds are maintained on the web by Demaine [11] and Sheffer [41]. For example, $\texttt{pm}(n) = O(10.07^n)$ means that any n-point set in the plane has at most $O(10.07^n)$ non-crossing perfect matchings, while $\texttt{pm}(n) = \Omega^*(3^n)$ means that there exists some n-point set that admits $\Omega^*(3^n)$ non-crossing perfect matchings (the O^* and Ω^* notation hides factors bounded by rational functions of n).

Abbrev.	Graph class	Lower bound	Upper bound
pg(n)	graphs	$\Omega(41.18^n)$ [2, 21]	$O(187.53^n)$ [37]
cf(n)	cycle-free graphs	$\Omega(12.26^n)$ [15]	$O(160.55^n)$ [24, 36]
pm(n)	perfect matchings	$\Omega^*(3^n)$ [21]	$O(10.07^n)$ [39]
st(n)	spanning trees	$\Omega(12.00^n)$ [15]	$O(141.7^n)$ [24, 36]
sc(n)	spanning cycles	$\Omega(4.64^n)$ [21]	$O(54.55^n)$ [38]
tr(n)	triangulations	$\Omega(8.65^n)$ [15]	$O(30^n)$ [36]

Table 2: Classes of non-crossing geometric graphs, current best lower and upper bounds.

Less studied are multiplicities of *weighted* non-crossing geometric graphs, where the weight of a geometric graph is its Euclidean length. The general question is: how many non-crossing graphs of a certain type (e.g., matchings) with *minimum* or *maximum* length can be realized on an n-point set in the plane? Table 3 displays some exponential lower bounds for several common types of weighted geometric graphs. The notation is analogous. Interestingly enough, no upper bounds better than those for the corresponding unweighted classes are known.

Abbrev.	Graph class	Lower bound
$\mathtt{pm}_{\min}(n)$	shortest perfect matchings	$\Omega(2^{n/4})$ [14]
$\mathtt{pm}_{\max}(n)$	longest perfect matchings	$\Omega(2^{n/4})$ [15]
$\mathtt{st}_{\min}(n)$	shortest spanning trees	$\Omega(2^{n/2})$ [14]
$\mathtt{st}_{\max}(n)$	longest spanning trees	$\Omega(2^n)$ [15]
$\mathtt{sc}_{\min}(n)$	shortest spanning cycles	$\Omega(2^{n/3})$ [15]
$\mathtt{sc}_{\max}(n)$	longest spanning cycles	$\Omega(2^{n/3})$ [15]

Table 3: Classes of *weighted* non-crossing geometric graphs: exponential lower bounds.

5 Non-crossing covering paths and trees

Every set of n points in the plane in general position admits a non-crossing spanning path (hence also tree) with $n-1$ edges: take for instance an x-monotone spanning path for the points. However, if we allow the addition of *Steiner* points (not necessarily in general position), then a non-crossing path or tree that contains the initial n points and has fewer than $n-1$ edges exists. A *covering path* (resp., *tree*) is a polygonal path (tree) that contains all the given points at vertices or in the interior of the edges (Fig. 1, right). For n points in general position, every covering path (resp., tree) requires at least $\lceil n/2 \rceil$ segments. A *minimum-link* covering path (tree) is one with the smallest number of segments (links).

There are arbitrarily large n-element point sets in general position for which any non-crossing covering path must have at least $5n/9 - O(1)$ segments [17]. Improving on the trivial upper bound of $n-1$, Gerbner and Keszegh [22] recently showed that every n-point set admits a covering path (and tree) with at most cn segments, for some positive constant $c < 1$. Establishing the minimum number of segments that suffice in a non-crossing covering path for n points in the plane remains as an open problem.

Welzl [12] asked what is the maximum integer $p(n)$ such that every set of n points in the plane contains a subset of size $p(n)$ that admits a covering path with $p(n)/2$ segments. Such a subset is called *perfect*. The current best lower bound, $p(n) = \Omega(\log n)$, follows from the Erdős-Szekeres theorem [19], but is quite far from the current (trivial) upper bound $p(n) = O(n)$.

6 Open problems

Interesting questions arise when counting certain types of subgraphs of a given fixed non-crossing geometric graph. For instance, suppose we have a fixed triangulation of an n-element point set. Kreveld et al. [28] asked what is the maximum number of convex polygons made up from edges of the given triangulation. Informally this question reads: How many potatoes are in a mesh? In a first partial answer they offered lower and upper bounds of $\Omega(1.5028^n)$ and $O(1.6181^n)$, respectively.

Similar questions can be posed about the maximum number of perfect matchings, spanning trees, or spanning cycles contained in a given triangulation on n points. For instance, Buchin et al. [9] have shown that any triangulation on n points contains at most $6^{n/4}$ perfect matchings. Partial answers to questions like these have been instrumental in deriving some of currently best upper bounds [15, 38] mentioned in Section 4.

We conclude with a few open problems:

1. Does there exist a constant c such that every geometric graph on n vertices and at least ckn edges contains k disjoint edges (i.e., a non-crossing matching with k edges)?

2. Does there exist a constant $c > 0$ such that every complete geometric graph on n vertices whose edges are colored by two colors contains a non-crossing monochromatic path of length cn? The current best lower bound is $\Omega(n^{2/3})$ [25].

3. What is the computational complexity of finding a longest non-crossing perfect matching (spanning tree, Hamiltonian path, or cycle)? Are these problems NP-hard, as suspected in [6]?

4. Can one efficiently compute a good approximation of the longest non-crossing Hamiltonian cycle of n points?

5. Does $\mathtt{sc}(S) < \mathtt{tr}(S)$ hold for every sufficiently large point set S, as conjectured in [38]?

6. Does there exist a constant $c < 1$ such that $\mathtt{pm}(S) = O(c^n \cdot \mathtt{tr}(S))$ for every point set S, as conjectured in [38]?

7. Is it possible to establish sharper upper bounds on the maximum multiplicity for weighted geometric graphs (better than those for the corresponding unweighted classes) [15]?

8. How many segments suffice in a non-crossing covering path for n points in the plane?

9. It is known that the minimum-link covering path problem is NP-complete for arbitrary point sets [5, 27]. Is the problem still NP-complete for points in general position and non-crossing paths [17]?

10. What is the size of the largest perfect subset that can be selected from any set of n points in the plane?

11. How many convex polygons can be made (at most) by using the edges from a fixed triangulation on n points? Can the gap between the bounds $\Omega(1.5028^n)$ and $O(1.6181^n)$ be closed?

References

[1] M. Abellanas, A. García, F. Hurtado, and J. Tejel, Caminos alternantes, *X Encuentros de Geometría Computacional* (in Spanish), Sevilla, 2003, pp. 7-12.

[2] O. Aichholzer, T. Hackl, B. Vogtenhuber, C. Huemer, F. Hurtado and H. Krasser, On the number of plane geometric graphs, *Graphs and Combinatorics* **23(1)** (2007), 67–84.

[3] M. Ajtai, V. Chvátal, M. Newborn and E. Szemerédi, Crossing-free subgraphs, *Annals of Discrete Mathematics* **12** (1982), 9–12.

[4] G. Araujo, A. Dumitrescu, F. Hurtado, M. Noy and J. Urrutia, On the chromatic number of some geometric type Kneser graphs, *Computational Geometry: Theory and Applications* **32** (2005), 59–69.

[5] E. M. Arkin, J. S. B. Mitchell and C. D. Piatko, Minimum-link watchman tours, *Information Processing Letters* **86** (2003), 203–207.

[6] N. Alon, S. Rajagopalan and S. Suri: Long non-crossing configurations in the plane, *Fundamenta Informaticae* **22** (1995), 385–394.

[7] S. Avital and H. Hanani, Graphs (in Hebrew), *Gilyonot Lematematika* **3** (1966), 2–8.

[8] P. Brass, G. Károlyi, P. Valtr, A Turán-type extremal theory of convex geometric graphs, in: Discrete and Computational Geometry, vol. 25 of Algorithms Combin., Springer, Berlin, 2003, pp. 275-300.

[9] K. Buchin, C. Knauer, K. Kriegel, A. Schulz, and R. Seidel, On the number of cycles in planar graphs, *Proc. 13th Annual International Conference on Computing and Combinatorics*, vol. 4598 of LNCS, Springer, 2007, pp. 97–107.

[10] J. Černý, Z. Dvořák, V. Jelínek, and J. Kára, Noncrossing Hamiltonian paths in geometric graphs, *Discrete Applied Mathematics* **155** (2007), 1096–1105.

[11] E. Demaine, Simple polygonizations, `http://erikdemaine.org/polygonization/` (version of October, 2012).

[12] E.D. Demaine and J. O'Rourke, Open problems from CCCG 2010, in *Proc. 23rd Canadian Conference on Computational Geometry*, 2011, Toronto, ON, pp. 153–156.

[13] A. Dumitrescu, On two lower bound constructions, *Proc. 11th Canadian Conference on Computational Geometry*, 1999, pp. 111–114.

[14] A. Dumitrescu, On the maximum multiplicity of some extreme geometric configurations in the plane, *Geombinatorics* **12(1)** (2002), 5–14.

[15] A. Dumitrescu, A. Schulz, A. Sheffer and Cs. D. Tóth, Bounds on the maximum multiplicity of some common geometric graphs, *Proc. 28th Symposium on Theoretical Aspects of Computer Science*, vol. 9 of *Leibniz International Proceedings in Informatics (LIPIcs)*, 2011, pp. 637–648.

[16] A. Dumitrescu and C. D. Tóth, Long non-crossing configurations in the plane, *Discrete and Computational Geometry* **44(4)** (2010), 727–752.

[17] A. Dumitrescu and Cs. D. Tóth, Covering paths for planar point sets, *Proc. 20th International Symposium on Graph Drawing (GD 2012)*, LNCS, Springer, to appear.

[18] D. Eppstein: Spanning trees and spanners, in J.-R. Sack and J. Urrutia (editors), *Handbook of Computational Geometry*, pages 425–461, Elsevier Science, Amsterdam, 2000.

[19] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Mathematica* **2** (1935), 463–470.

[20] A. P. Figueroa, A note on a theorem of Perles concerning non-crossing paths in convex geometric graphs, *Computational Geometry: Theory and Applications* **42** (2009) 90-91.

[21] A. García, M. Noy and A. Tejel, Lower bounds on the number of crossing-free subgraphs of K_N, *Computational Geometry: Theory and Applications* **16(4)** (2000), 211–221.

[22] D. Gerbner and B. Keszegh, Non-crossing covering paths for planar point sets, manuscript, September 2012.

[23] P. Gritzmann, B. Mohar, J. Pach, and R. Pollack, Embedding a planar triangulation with vertices at specified points, *American Mathematical Monthly* **98** (1991), 165–166.

[24] M. Hoffmann, M. Sharir, A. Schulz, A. Sheffer, Cs. D. Tóth, and E. Welzl, Counting plane graphs: flippability and its applications, *Thirty Essays on Geometric Graph Theory* (J. Pach, ed.), vol. 29 of Algorithms and Combinatorics, Springer, 2012, to appear. Also at `arxiv.org/abs/1012.0591`.

[25] G. Károlyi, J. Pach, and G. Tóth, Ramsey-type results for geometric graphs, I, *Discrete & Computational Geometry* **18(3)** (1997), 247–255.

[26] G. Károlyi, J. Pach, G. Tóth, and P. Valtr, Ramsey-type results for geometric graphs, II, *Discrete & Computational Geometry* **20(3)** (1998), 375–388.

[27] E. Kranakis, D. Krizanc and L. Meertens, Link length of rectilinear Hamiltonian tours in grids, *Ars Combinatoria* **38** (1994), 177–192.

[28] M. van Kreveld, M. Löffler, and J. Pach, How many potatoes are in a mesh? *Proc. 23rd International Symposium on. Algorithms and Computation (ISAAC 2012)*, LNCS, Springer, 2012, to appear. Also at `arxiv.org/abs/1209.3954`.

[29] Y. Kupitz, Extremal problems in combinatorial geometry, *Aarhus University Lecture Notes Series*, **53** (1979), Aarhus University, Denmark.

[30] J. Kynčl, J. Pach, and G. Tóth, Long alternating paths in bicolored point sets, *Discrete Mathematics* **308(19)** (2008), 4315–4321.

[31] C. Merino, G. Salazar, and J. Urrutia, On the length of the longest alternating paths for multicolored point set, *Discrete Mathematics* **306** (2006) 1791-1797.

[32] J. S. B. Mitchell: Geometric shortest paths and network optimization, in J.-R. Sack and J. Urrutia (editors), *Handbook of Computational Geometry*, pp. 633–701, Elsevier, Amsterdam, 2000.

[33] M. Newborn and W. O. J. Moser, Optimal crossing-free Hamiltonian circuit drawings of K_n, *Journal of Combinatorial Theory Ser. B* **29** (1980), 13–26.

[34] A. Razen, J. Snoeyink, and E. Welzl, Number of crossing-free geometric graphs vs. triangulations, *Electronic Notes in Discrete Mathematics* **31** (2008), 195–200.

[35] F. Santos and R. Seidel, A better upper bound on the number of triangulations of a planar point set, *Journal of Combinatorial Theory Ser. A* **102(1)** (2003), 186–193.

[36] M. Sharir and A. Sheffer, Counting triangulations of planar point sets, *Electronic Journal of Combinatorics* **18** (2011), P70.

[37] M. Sharir and A. Sheffer, Counting plane graphs: cross-graph charging schemes, *Proc. 20th International Symposium on Graph Drawing (GD 2012)*, LNCS, Springer, to appear.

[38] M. Sharir, A. Sheffer, and E. Welzl, Counting plane graphs: perfect matchings, spanning cycles, and Kasteleyn's technique, *Proc. 28th ACM Symp. on Computational Geometry* (2012), 189–198.

[39] M. Sharir and E. Welzl, On the number of crossing-free matchings, cycles, and partitions, *SIAM Journal on Computing* **36(3)** (2006), 695–720.

[40] M. Sharir and E. Welzl, Random triangulations of planar point sets, *Proc. 22nd Annual ACM-SIAM Symposium on Computational Geometry*, ACM Press, 2006, pp. 273–281.

[41] A. Sheffer, Numbers of plane graphs, `http://www.cs.tau.ac.il/~sheffera/counting/PlaneGraphs.html` (version of October, 2012).

[42] G. Tóth, Note on geometric graphs, *Journal of Combinatorial Theory, Ser. A* **89** (2000), 126–132.